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ELEMENTS OF
TRIGONOMETRY

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ELEMENTS OF TRIGONOMETRY

Plane and Spherical with Applications

BY

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McGRAW-HILL BOOK COMPANY, INC.

NEW YORK AND LONDON

ELEMENTS OF TRIGONOMETRY

PRINTED IN THE UNITED STATES OF AMERICA

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THE MAPLE PRESS COMPANY, YORK, PA.

PREFACE

In the preparation of this text for use in the secondary school, the authors have attempted to tone down Kells, Kern, and Bland's "Plane and Spherical Trigonometry" to the level of students in the third and fourth years of the high school. While they have retained the features of the book which have appealed strongly to teachers and students, particularly the variety of problems and naval and military applications, they have introduced the following changes:

1. After an introductory chapter concerning the trigonometric functions of an acute angle, the student is introduced without delay to the solution of the right triangle with nautical applications and an elementry treatment of vectors and rectangular components. This early presentation of the right triangle allows for greater exposure to this part of the trigonometry. The teacher must not attempt to complete this second chapter before going on to the further topics, but rather will use part of it for further practice and review while he develops the content of the chapters that follow: Fundamental Relations among the Functions, General Definitions of the Functions, The Radian, The Mil and Graphs, and General Formulas.

2. The solution of the oblique triangle is presented in Chap. 7, in which the necessary formulas are applied to the triangle as they are developed. Thus, the student gets a complete picture of the solution of the oblique triangle. If a teacher so desires, he may begin the work of this chapter before he completes the content of Chaps. 3, 4, 5, and 6, again offering the students longer exposure to the applications involved.

3. The development of the trigonometric functions of the general angle in Chap. 4 has been simplified through a rearrangement of the topics which gives it more unity and lends itself to easier teaching.

Chapter 8, which completes the plane trigonometry portion of the book, presents an elementary treatment of the inverse trigonometric functions that is more than sufficient for any

course in high school trigonometry. The teacher need not attempt to cover all of it.

The spherical trigonometry is presented in Chaps. 9, 10, 11, and 12 by a gradual development from the right spherical triangle with elementary applications to the terrestrial sphere leading to the oblique spherical triangle followed by the treatment of the celestial sphere. This portion of the book may be used for a complete course in spherical trigonometry; or, from it, a teacher may select enough to make up a brief unit which he may wish to teach as part of a semester's work in trigonometry.

Chapter 13 contains a complete treatment of the topic of logarithms as it is used in connection with the trigonometry. Although pupils in most high schools come to the trigonometry class with a knowledge of logarithms, this chapter is retained so that the class will have it for reference and for review.

A great deal of emphasis is being placed these days on the use of the slide rule in courses in high school mathematics. Chapter 14 presents a rather complete treatment of the slide rule and its use in trigonometric problems. Teachers and students who otherwise must depend on supplementary pamphlets will welcome this feature of the book.

The explanatory matter, which is so readable and easily understood, the variety of illustrations which give a certain reality to the problems, the abundance of both exercises and problems, and the miscellaneous review exercises will strongly recommend the book to both teachers and students.

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ANNAPOLIS, MD.,
NEW YORK CITY,
August, 1943.

CONTENTS

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
α	Alpha	ι	Iota	ρ	Rho
β	Beta	κ	Kappa	σ s	Sigma
γ	Gamma	λ	Lambda	τ	Tau
δ	Delta	μ	Mu	υ	Upsilon
ϵ	Epsilon	ν	Nu	φ	Phi
ζ	Zeta	ξ	Xi	χ	Chi
η	Eta	o	Omicron	ψ	Psi
θ	Theta	π	Pi	ω	Omega

LIST OF SYMBOLS

$\equiv$, read *is idential with.*
$\neq$, read *is not equal to.*
$<$, read *is less than*
$>$, read *is greater than.*
$\leqq$, read *is less than or equal to.*
$\geqq$, read *is greater than or equal to.*
$\stackrel{\circ}{=}$, read *contains the same number of degrees as.*
(x, y), read *point whose coordinates are x and y.*

ELEMENTS OF TRIGONOMETRY

CHAPTER 1

TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

1-1. Introduction. A cadet who was 6 ft. tall found that his shadow was 3 ft. long (see Fig. 1-1). He argued that since his height was twice the length of his shadow, the height of a near-by flagpole must be twice the length of its shadow. He then measured the shadow of the flagpole and found it was 7 ft.

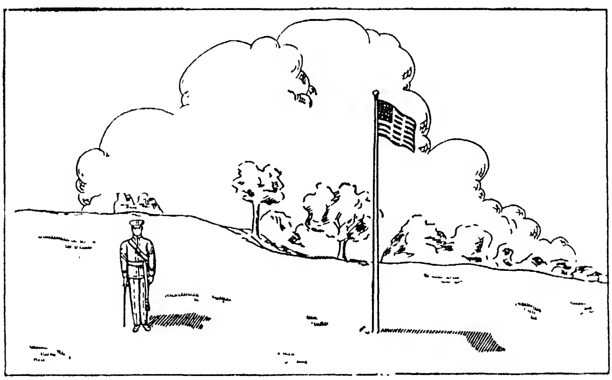


FIG. 1-1.

long. He concluded that the height of the flagpole was twice the length of its shadow, or 2×7 ft. $= 14$ ft. In other words, by observing that the ratio of the height of a certain right triangle to its base was $\frac{2}{1}$, he found the height of a flagpole without measuring it.

This illustration really involves two right triangles. Lines representing the cadet and his shadow are the legs of one triangle, the hypotenuse being the ray of light from the top of the cadet's head to the outer end of the shadow. Likewise, lines representing the flagpole and its shadow are the legs of the other triangle,

the hypotenuse being the ray of light from the top of the pole to the outer end of its shadow. Furthermore, since the shadows are being considered at the same time of the day, the angle formed by each ray of light with the ground is the same. Hence, the two right triangles are similar; and you know from your study of plane geometry that the corresponding sides of similar triangles are in proportion.

This is a very elementary illustration of what navigators, surveyors, engineers, and others do with trigonometry. By applying the complete theory of the ratios of the sides of a right triangle (that is, trigonometry) to data obtained by measurements, they find inaccessible heights of mountains and distances through them; distances across lakes, rivers, and inaccessible swamps; boundaries of fields and countries, and positions at sea. Engineers use trigonometry every day in their work of constructing large buildings, bridges, and roads; astronomers use it to determine the time by which clocks are regulated; surveyors use it constantly to find all sorts of heights, distances, and directions; and navigators use it to compute latitude, longitude, and course at sea.

Trigonometry has other very important uses. The ratios of the sides of right triangles are capable of describing phenomena of a periodic nature such as the to-and-fro motion of a pendulum and the motion of waves. Consequently, they play an important part in the theory of light and sound, in electrical theory, in wave analysis, and in all investigations dealing with phenomena of a vibratory character. Hence, although most of the problems stated in this book to illustrate practical phases of trigonometry deal with heights of inaccessible objects and distances, a large number of exercises will help to familiarize the student with a class of functions of great importance in more advanced mathematical theory.

1-2. Ratio. At the very base of trigonometry lies the idea of ratio. The ratio of number a to number b is expressed as the quotient of a divided by b, that is, $\frac{a}{b}$. Thus, the ratio of 3 to 5 is represented by the fraction $\frac{3}{5}$; the ratio of 12 to 4 is $\frac{12}{4}$ or 3. The ratio of two line segments is the ratio of the length of one segment to the length of the other expressed in the same unit.

The ratio of a line segment 1 unit long to another 2 units long is $\frac{1}{2}$, whether the lengths be expressed in miles or in feet.

One of the main reasons for the usefulness of trigonometry is that it furnishes a method of finding ratios associated with angles. One gets some idea of the importance of a knowledge of these ratios by considering the usefulness of models of machines, of blueprints of buildings, and of various kinds of maps. The plane angle made by two straight lines in the model is the same as the angle made by the corresponding lines in the actual structure; therefore the ratios associated with the angles in the model will be the same as those in the corresponding angles in the structure represented. Thus the angles made by corresponding lines in the similar diamonds represented in Fig. 1-2 are equal. The cadet mentioned in Art. 1-1 found the height of the flagpole by using the ratio of the length of an object to that of its shadow. A traveler can find distances approximately by using the fact that map distances have the same ratio as actual distances.

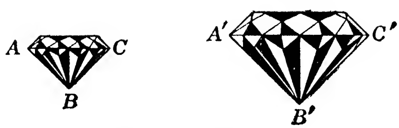


FIG. 1-2.

1-3. Definition of Function. If every value of a variable x within a certain interval is associated with a value of another variable y in such a way that when x is given, y is determined, then y is said to be a **function** of x. For example, in the equality $y = 2x + 1$,

$$\text{if } x = 4, 3, 2, 1, 0, -1, -2, -3, \text{ etc.}$$
$$\text{then } y = 9, 7, 5, 3, 1, -1, -3, -5, \text{ etc.}$$

Hence, y is a function of x.

Likewise, the area of a square is a function of its side, since when the side is given, the area is determined; the distance covered by a car running at a constant speed is a function of the time, since, if the time is given, the distance is determined. Later we shall find that certain ratios of lengths of line segments are functions of angles.

1-4. The tangent, the sine, and the cosine. Consider an acute angle such as angle A in Fig. 1-3. From any point B on one side of the angle drop a perpendicular to the other side,

meeting it in C, and consider the ratio CB/AC. The value of this ratio is determined when the angle is given. Let $B'C'$ and $B''C''$ represent any other lines drawn from points B' and B'' on one side of angle A perpendicular to the other side and meeting it in C' and C'', respectively. Then the triangles ABC, $AB'C'$, and $AB''C''$ are similar, since they are right triangles having an acute angle in common. Since the corresponding sides of similar triangles are in proportion,

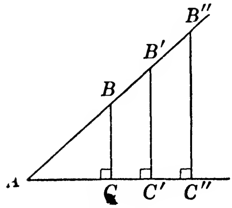


FIG. 1-3.

$$\frac{CB}{AC} = \frac{C'B'}{AC'} = \frac{C''B''}{AC''}. \tag{1}$$

Thus the value of the ratio CB/AC is determined when an acute angle is given. If angle A were to increase or decrease, the lengths of the line segments would vary and the ratios would be different from those in (1); but for the new angle the three ratios would be equal. In accordance with the definition in Art. 1-3, *this ratio is a function of the acute angle.* The ratio CB/AC in Fig. 1-3 is named the **tangent** of angle A, and we write

$$\tan A = \frac{CB}{AC}. \tag{2}$$

Also, two acute angles that have the same tangent are equal. Let A and A' in Fig. 1-4 be two angles such that

$$\tan A = \tan A'. \tag{3}$$

Construct the right triangles shown in Fig. 1-4. Then, from (3)

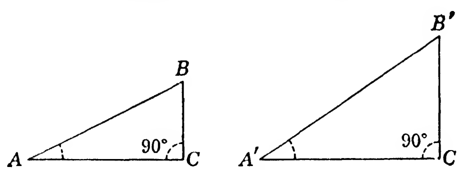


FIG. 1-4.

and the definition (2),

$$\frac{CB}{AC} = \tan A = \tan A' = \frac{C'B'}{A'C'}. \tag{4}$$

Hence the two triangles in Fig. 1-4 are similar, having an angle

(90°) of one equal to an angle of the other and the including sides proportional. Therefore angle A and angle A', being corresponding angles of similar triangles, are equal.

For convenience, we shall indicate that an angle is a right angle by drawing a small square at its vertex. Thus the small square at C in Fig. 1-5 shows that angle C is a right angle.

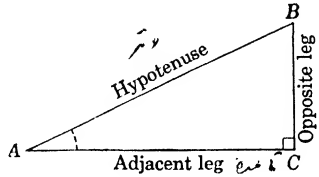


FIG. 1-5.

Two other ratios, besides the tangent of an angle, are very important. The ratio CB/AB in Fig. 1-5 is called the **sine** of angle A, and the ratio AC/AB is called the **cosine** of angle A. Using the abbreviations **cos** for **cosine** and **sin** for **sine**, we have from Fig. 1-5.

$$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}}, \qquad \cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}},$$

$$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}}. \tag{5}$$

These ratios are called trigonometric functions. By using the same line of reasoning applied in the case of the tangent, we can show that *the value of each of the three trigonometric functions of an acute angle is determined when the acute angle is given.* Furthermore, it can be shown that *if the value of any one of the three trigonometric functions of an acute angle is equal to the value of the same function of a second acute angle, the two acute angles are equal.*

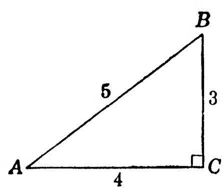


FIG. 1-6.

Example 1. Find the values of the three trigonometric functions of an angle A if its sine is $\frac{3}{5}$.

Solution. Draw a right triangle having its hypotenuse 5 units long and one leg 3 units long (see Fig. 1-6). The acute angle opposite the 3-unit leg is angle A, since its sine is $\frac{3}{5}$. Also, the side $AC = \sqrt{25 - 9} = 4$. Then, from Fig. 1-6, we read in accordance with the definitions (5)

$$\sin A = \tfrac{3}{5},$$
$$\cos A = \tfrac{4}{5},$$
$$\tan A = \tfrac{3}{4}.$$

Example 2. A surveyor wishing to find the height of a lighthouse measures the angle A at a point 120 ft. from its base. His findings are represented in Fig. 1-7, where $\tan A = \frac{2}{3}$. What is the height of the lighthouse?

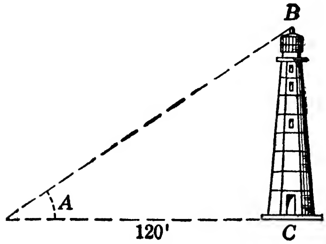


FIG. 1-7.

Solution. From triangle ABC we read

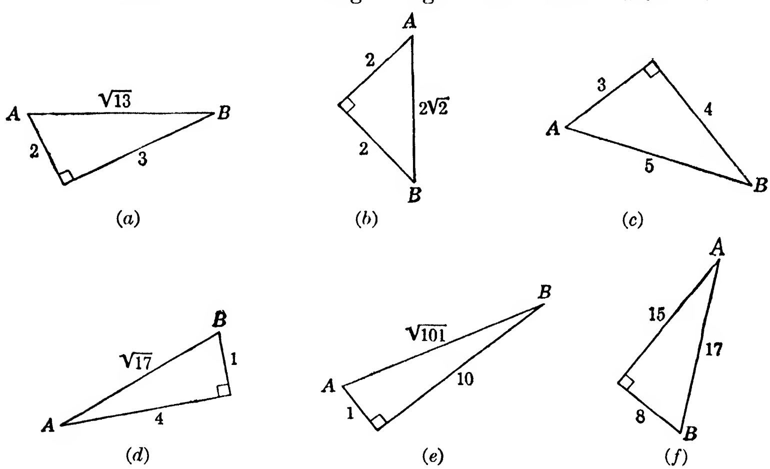
$$\tan A = \frac{CB}{AC}, \quad \text{or} \quad \tan A = \frac{CB}{120}.$$

Solving this equation for CB and replacing $\tan A$ by its value $\frac{2}{3}$, we obtain

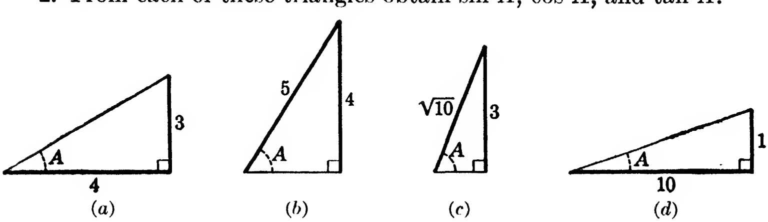
$$CB = 120 \tan A = 120(\tfrac{2}{3}) = \mathbf{80\ ft.}^*$$

EXERCISES 1-1

1. From each of the following triangles read $\tan A$ and $\tan B$.



2. From each of these triangles obtain $\sin A$, $\cos A$, and $\tan A$.



* Throughout this book the answers to illustrative examples will be printed in **boldface** characters.

3. If $\sin A = \frac{5}{13}$, find $\cos A$ and $\tan A$.

4. If $\cos A = \frac{7}{25}$, find $\sin A$ and $\tan A$.

5. If $\tan A = \frac{8}{15}$, find $\sin A$ and $\cos A$.

6. If $\sin A = \frac{8}{17}$, find $\cos A$ and $\tan A$.

7. If $\cos A = \frac{24}{25}$, find $\sin A$ and $\tan A$.

8. If $\cos A = \frac{15}{17}$, find $\sin A$ and $\tan A$.

9. If $\sin A = \dfrac{1}{\sqrt{2}}$, show that $\sin A = \cos A$.

10. For angle A in Exercise 2*a*, show that

(*a*) $\sin A \cos A = \frac{12}{25}$. (*b*) $\dfrac{\sin A}{\cos A} \tan A = \frac{9}{16}$.

(*c*) $(\sin A)^2 + (\cos A)^2 = 1$. (*d*) $\dfrac{1}{(\cos A)^2} - (\tan A)^2 = 1$.

11. An observer at A (see Fig. 1-8), 1110 ft. from and on a level with the base of the Washington Monument, sights its top and finds that the angle A is such that $\tan A = \frac{1}{2}$. Find the height of the monument.

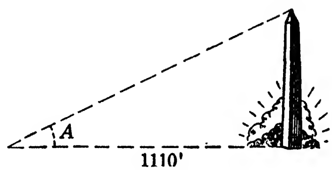


FIG. 1-8.

12. A base line AC 350 ft. in length is laid along one bank of a river. On the opposite bank a point B is located so that CB is perpendicular to AC. The tangent of the angle CAB is then measured and found to be $\frac{16}{5}$. Find the width of the river.

13. Figure 1-9 represents a ladder leaning against the side of a house. If the ladder is 36 ft. long and $\cos A = \frac{1}{4}$, how far is the foot of the ladder from the house?

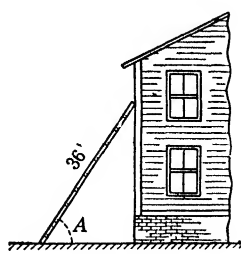


FIG. 1-9.

14. The length of string between a kite and a point on the ground is 225 ft. If the string is straight and makes with the level ground an angle whose tangent is $\frac{15}{8}$, how high is the kite?

15. Figure 1-10 shows the relative positions of a point O and two oil wells, A and C, 300 ft. apart. An observer at O finds that the sine of angle AOC is $\frac{1}{5}$. What is his distance from the well at A?

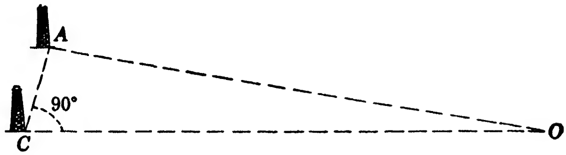


FIG. 1-10.

1-5. The cotangent, the secant, and the cosecant. Besides the three ratios of pairs of sides of a right triangle in Art. 1-4, there are three others obtained by writing their reciprocals. The reciprocals of tan A, cos A, and sin A are called, respectively, **cotangent** A, **secant** A, and **cosecant** A, and are represented by **cot** A, **sec** A, and **csc** A.

Referring to the right triangle in Fig. 1-11, we make the following definitions:

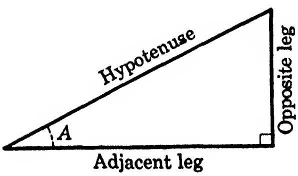


FIG. 1-11.

$$\left.\begin{aligned} \cot A &= \frac{\text{adjacent leg}}{\text{opposite leg}}, \\ \sec A &= \frac{\text{hypotenuse}}{\text{adjacent leg}}, \\ \csc A &= \frac{\text{hypotenuse}}{\text{opposite leg}}. \end{aligned}\right\} \qquad (6)$$

Just as before, the value of each trigonometric function is determined when the acute angle is given; and if the value of any one of the six trigonometric functions of an acute angle is equal to the value of the same function of a second acute angle, the two acute angles are equal.

Since $y/x = 1 \div (x/y)$, it appears from the definitions (5) and (6) and Fig. 1-12 that

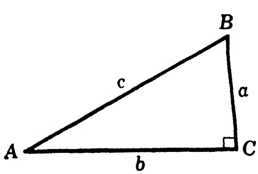


FIG. 1-12.

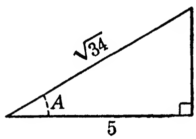
$$\left.\begin{aligned} \csc A &= \frac{c}{a} = \frac{1}{a/c} = \frac{1}{\sin A}, \\ \sec A &= \frac{c}{b} = \frac{1}{b/c} = \frac{1}{\cos A}, \\ \cot A &= \frac{b}{a} = \frac{1}{a/b} = \frac{1}{\tan A}. \end{aligned}\right\} \qquad (7)$$

It will be well for the student to think of csc A, sec A, and cot A as reciprocals of sin A, cos A, and tan A, respectively; *thus, to find* csc A, *think of the fraction for* sin A *and then write its reciprocal.*

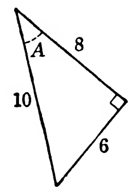
In labeling the parts of right triangle ABC, it is customary to let C refer to the vertex of the right angle and A and B to the other vertices. The hypotenuse is then called c, and the legs opposite A and B are called a and b respectively.

EXERCISES 1-2

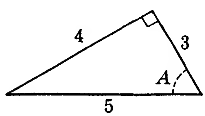
1. In each of the following triangles write the six trigonometric functions of angle A.



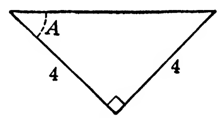
(*a*)



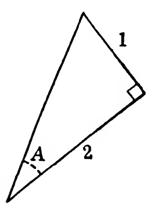
(*b*)



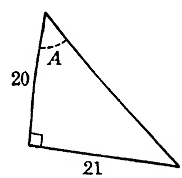
(*c*)



(*d*)



(*e*)



(*f*)

2. The sides of a right triangle are 5, 12, and 13, respectively. Read the values of the trigonometric functions of the angle opposite the 5-unit leg. Also read the functions of the angle opposite the 12-unit leg.

3. Find the values of all the trigonometric functions of an acute angle having (*a*) its sine equal to $\frac{4}{5}$; (*b*) its tangent equal to $\frac{8}{15}$; (*c*) its cosine equal to $\frac{1}{2}$.

4. If $\sin A = \frac{6}{7}$, find the value of

(*a*) $(\sin A)^2 + (\cos A)^2$. (*b*) $(\csc A)^2 - (\cot A)^2$.

5. Given that $\sin D = \frac{4}{5}$, $\tan E = \frac{5}{12}$, $\cos F = \frac{8}{17}$, $\cot G = \frac{24}{7}$, show that the following equations are true:

(a) $(\cos D)^2 \sec G \cos E = \frac{9}{26}$.
(b) $(\csc D)^2 \cot F \cot E = 2$.
(c) $\sec E \tan F \cot G \sin G \tan D = \frac{13}{5}$.
(d) $\sin D \csc E \sec G \cos E = 2$.
(e) $\csc D \cot F \csc G \cos E = \frac{200}{91}$.

6. The relative positions of the point A at the bow of a ship 300 ft. long, C at its stern, and B on a near-by submarine are shown in Fig. 1-13.

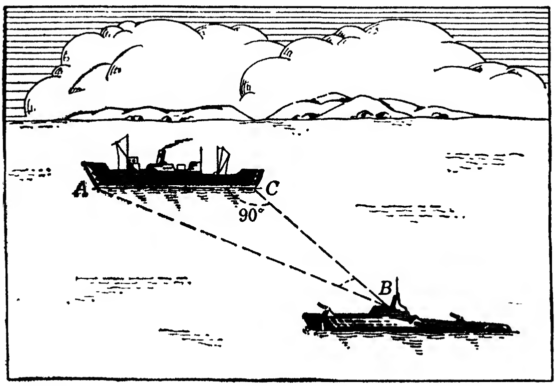


Fig. 1-13.

If the tangent of angle ABC is $\frac{5}{3}$ and angle ACB is 90°, about how far is the submarine from the ship?

7. The central pole of a circular tent is 30 ft. high and is fastened at the top by ropes to stakes set in the ground. Each rope makes an angle A with the ground such that $\csc A = \frac{3}{2}$. Find the length of each rope.

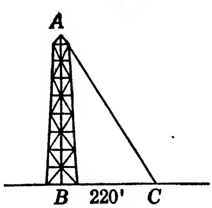


Fig. 1-14.

8. Figure 1-14 represents a radio tower. AC is a cable anchored at point C on a level with the base of the tower. The angle C made by the cable with the horizontal is such that $\sec C = \frac{9}{5}$. If the distance from C to the center B of the base is 220 ft., find the length of the cable.

9. The hypotenuse of a right triangle is 800 ft., and $\sin A = \frac{12}{13}$. Find the legs of the triangle.

10. The following data refer to right triangles. In each case find the unknown sides:

(a) $c = 520$, $\sin A = \frac{3}{5}$. (b) $a = 880$, $\cos A = \frac{8}{17}$.
(c) $b = 34$, $\tan B = \frac{1}{2}$. (d) $c = 250$, $\cot B = \frac{12}{5}$.
(e) $a = 173$, $\csc B = 3$. (f) $b = 284$, $\sin B = \frac{1}{3}$.

1-6. Trigonometric functions of 45°, 30°, 60°. In Fig. 1-15 you see a square with sides one unit in length. Diagonal AB makes an angle of 45° with the side, and is $\sqrt{1^2 + 1^2}$ or $\sqrt{2}$ units long. From triangle ABC, in accordance with definitions (5) and (6), we read

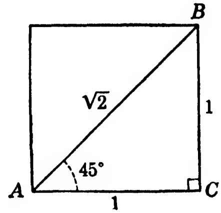


FIG. 1-15.

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = 0.7071,$$
$$\cos 45^\circ = \frac{1}{\sqrt{2}} = 0.7071,$$
$$\tan 45^\circ = 1 = 1.0000,$$
$$\cot 45^\circ = 1 = 1.0000,$$
$$\sec 45^\circ = \sqrt{2} = 1.4142,$$
$$\csc 45^\circ = \sqrt{2} = 1.4142.$$

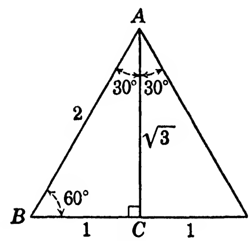


FIG. 1-16.

In Fig. 1-16, you see an equilateral triangle with sides two units in length. From your study of plane geometry you know that AC, the bisector of angle A, is also the median and the altitude on the opposite side. Hence, triangle ABC is a right triangle and side BC is one unit long. $AC = \sqrt{2^2 - 1^2}$ or $\sqrt{3}$. In accordance with the definitions of the functions we obtain from triangle ABC

$$\sin 30^\circ = \tfrac{1}{2} = 0.5000, \qquad \cot 30^\circ = \sqrt{3} = 1.7321,$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2} = 0.8660, \qquad \sec 30^\circ = \frac{2}{\sqrt{3}} = 1.1547,$$
$$\tan 30^\circ = \frac{1}{\sqrt{3}} = 0.5774, \qquad \csc 30^\circ = 2 = 2.0000.$$

Also,
$$\sin 60^\circ = \frac{\sqrt{3}}{2} = 0.8660, \qquad \cot 60^\circ = \frac{1}{\sqrt{3}} = 0.5774,$$
$$\cos 60^\circ = \tfrac{1}{2} = 0.5000, \qquad \sec 60^\circ = 2 = 2.0000,$$
$$\tan 60^\circ = \sqrt{3} = 1.7321, \qquad \csc 60^\circ = \frac{2}{\sqrt{3}} = 1.1547.$$

1-7. Trigonometric functions of 0° and 90°. In Fig. 1-17, if angle A is regarded as a very small angle and approaching the value 0°, the opposite side BC approaches zero units in length.

At the same time, AB becomes equal in length to AC. The values of the functions of 0° are obtained in accordance with the definitions in (5) and (6). Since division by zero is excluded from algebraic operations, it appears that csc 0° and cot 0° are undefined. Nevertheless, we write csc 0° = ∞ and cot 0° = ∞ and mean by these symbols that, as an acute angle θ varies and approaches zero as a limit, the values of csc θ and cot θ vary and become greater and greater without limit. Hence, from Fig. 1-17, we write

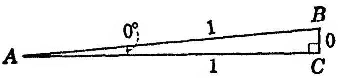


FIG. 1-17.

$$\sin 0° = 0, \qquad \csc 0° = \infty,$$
$$\cos 0° = 1, \qquad \sec 0° = 1,$$
$$\tan 0° = 0, \qquad \cot 0° = \infty.$$

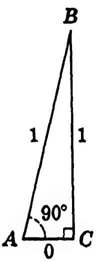


FIG. 1-18.

Similarly, from Fig. 1-18, we write

$$\sin 90° = 1, \qquad \csc 90° = 1,$$
$$\cos 90° = 0, \qquad \sec 90° = \infty,$$
$$\tan 90° = \infty, \qquad \cot 90° = 0.$$

EXERCISES 1-3

1. Draw a right triangle, one of whose acute angles is 30°. Assign appropriate lengths to the sides of this right triangle, and from it read the values of the trigonometric functions of 30° and of 60°.

2. Find approximately the values of the trigonometric functions of 1′ by reading them from Fig. 1-19. From this same figure read the approximate values of the trigonometric functions of 89°59′.

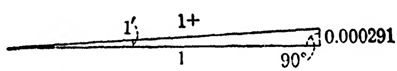


FIG. 1-19.

3. Draw a triangle from which may be read the values of the trigonometric functions of an angle A whose sine is $\frac{9}{41}$. From this figure read the values of the trigonometric functions of A and of $90° - A$.

4. If sec $A = 2$, write the trigonometric functions of A.

5. If tan $A = 1$, write the trigonometric functions of A.

6. Prove that $\cos 60° = 2 \cos^2 30° - 1$.

7. Prove that $\tan 30° = \dfrac{\sec 60°}{(\sec 60° + 1) \csc 60°}$.

8. Find the values of each of the following:

(a) $\tan 30° \sin 60° \sec 30° \cot 45°$.
(b) $\csc 45° \sin 90° \tan 60° \cos 0°$.
(c) $\cos 45° \csc 45° - \tan 45° \tan 0°$.
(d) $\sin 30° \sin 45° \cos 0° \csc 60° \cot 60°$.

9. Show that:

(a) $\sin 90° = \sin 30° \cos 60° + \cos 30° \sin 60°$.
(b) $\cos 30° = \cos 60° \cos 30° + \sin 60° \sin 30°$.
(c) $\sin 30° = \sin 60° \cos 30° - \cos 60° \sin 30°$.

10. If $\tan A = \tan 45° \cos 30° \tan 60°$, find the trigonometric functions of A.

11. That the formulas

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

are true for all values of A and B will be proved in Chap. 6. In these formulas substitute 45° for A and 30° for B and evaluate the resulting right-hand members to obtain sin 75° and cos 15°, respectively.

12. A tree stands at a certain distance from a straight road on which two milestones are located. The tree was observed from each milestone, and the angles between the lines of sight and the road were found to be 30° and 90°, respectively. Find the distance from the tree to the road.

13. The ladder leaning against the wall in Fig. 1-20 is 45 ft. long. If it makes an angle of 60° with the horizontal, how far is the foot of the ladder from the wall?

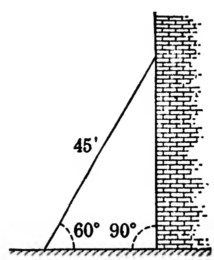
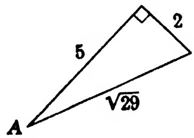


FIG. 1-20.

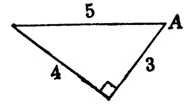
14. A farmer wishes to fence a field in the form of a right triangle. If one angle of the triangle is 45° and the hypotenuse is 200 yd., find the amount of fencing needed.

MISCELLANEOUS EXERCISES 1-4

1. In each of these triangles read the six trigonometric functions of angle A.



(a)



(b)

2. If sec $A = \frac{17}{8}$, find sin A, cos A, and cot A.

3. If sin $A = \frac{3}{5}$, show that

(a) $\cos A \cot A = \frac{16}{15}$. (b) $\sin^2 A + \cos^2 A = 1$.
(c) $1 + \tan^2 A = \sec^2 A$. (d) $1 + \cot^2 A = \csc^2 A$.

4. Find the values of the trigonometric functions of an acute angle having (a) its sine equal to $\frac{4}{5}$; (b) its tangent equal to $\frac{8}{15}$; (c) its cosine equal to $\frac{12}{13}$.

5. If sin $B = \frac{24}{25}$, find the value of

(a) $2 \sin B \cos B$. (b) $\cos^2 B - \sin^2 B$.

6. If $\sin A = \dfrac{1}{\sqrt{2}}$, find sin $2A$ by means of the formula (to be derived later)

$$\sin 2A = 2 \sin A \cos A.$$

7. If sin $A = \frac{1}{2}$ and cos $B = \frac{3}{4}$, find the value of sin $(A + B)$ by means of the formula (to be derived later)

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

8. The base of an isosceles triangle is 30 units, and each of its base angles has $\frac{5}{13}$ as the value of its cosine. Find the lengths of the altitudes and of the sides of the triangle.

9. For a certain triangle ABC, sin $A = \frac{12}{13}$, tan $B = \frac{15}{8}$, and the altitude to side AB is 60 units. Find the lengths of the sides and of the altitudes of the triangle.

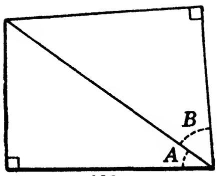


FIG. 1-21.

10. Find all unknown line segments in Fig. 1-21 if sin $A = \frac{3}{5}$, tan $B = \frac{6}{5}$.

11. Find all unknown sides in radical form and all unknown angles in Fig. 1-22.

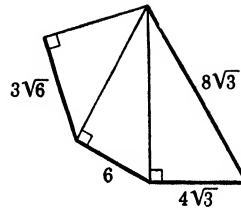


FIG. 1-22.

12. If, in Fig. 1-23, $\tan A = \frac{9}{4}$ and $\sec B = \frac{5}{3}$, find x, y, and z.

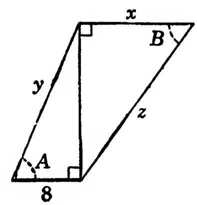


FIG. 1-23.

13. If, in Fig. 1-24, $\tan A = \frac{5}{12}$ and $\tan B = \frac{3}{4}$, find x, y, and z.

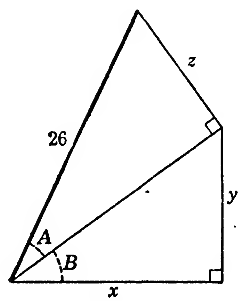


FIG. 1-24.

14. If, in Fig. 1-25, $\sin A = \frac{3}{5}$ and $\tan B = \frac{5}{12}$, find s, t, w, x, y, and z.

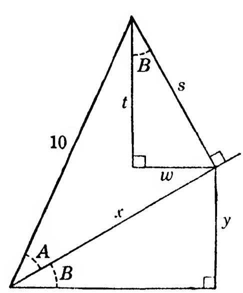


FIG. 1-25.

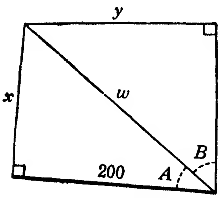


FIG. 1-26.

15. If, in Fig. 1-26, $\sin A = \frac{3}{5}$ and $\tan B = \frac{8}{15}$, find the lengths of all the line segments.

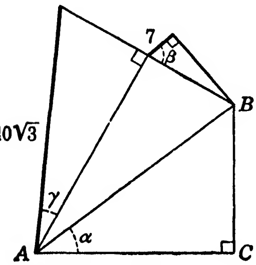


FIG. 1-27.

16. In Fig. 1-27 $\tan \alpha = \frac{3}{4}$, $\sin \gamma = \frac{1}{2}$, and $\sin \beta = \frac{24}{25}$. Compute the lengths of the sides of triangle ABC, and write the trigonometric functions of angle ABC.

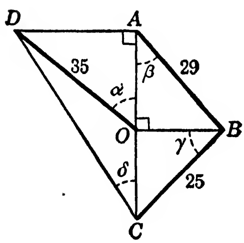


FIG. 1-28.

17. If, in Fig. 1-28, $\csc \alpha = \frac{5}{4}$, $AB = 29$ units, $BC = 25$ units, and $OD = 35$ units, find the lengths of all line segments in the figure, and write the values of the trigonometric functions of β, of γ, and of δ. Also find the length of the perpendicular from O to the line DC.

18. At a point A in a horizontal plane through the base of a flagpole the angle of elevation of its top is 35°. If the flagpole is 40 ft. high, find the distance from A to the pole ($\sin 35° = 0.574$, $\cos 35° = 0.819$, $\tan 35° = 0.700$).

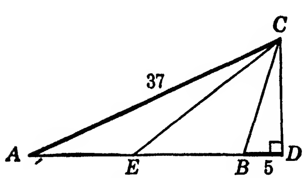


FIG. 1-29.

19. In Fig. 1-29 CE is the median to side AB of the triangle ABC, $\tan A = \frac{12}{35}$, $AC = 37$ units, and $BD = 5$ units. Find the lengths of all line segments in the figure, and write the trigonometric functions of angle DCE.

20. If, in Fig. 1-30, $\sin\theta = \frac{3}{5}$, $\cos\varphi = \frac{3}{5}$, $AB = 20$ ft., and $CA = 16$ ft., find the lengths of all line segments in the figure. Also find the values of the trigonometric functions of angle AED.

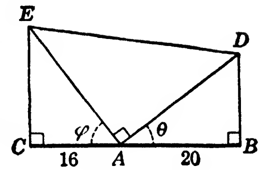


FIG. 1-30.

21. In Fig. 1-31 ABC is an arc of a circle with center at O. Prove that angle DAB is 15°. Compute the lengths DB, DA, and AB in radical form, and then write the trigonometric functions of 15°.

22. Construct a figure like Fig. 1-31 but with 45° in place of 30°. Use the figure to find the trigonometric functions of $22\frac{1}{2}°$.

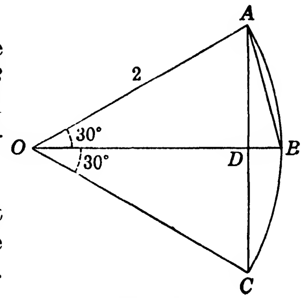


FIG. 1-31.

23. Prove that the area K of a right triangle (see Fig. 1-32) may be expressed by

$$K = \tfrac{1}{2}a \times b = \tfrac{1}{2}ac\cos A = \tfrac{1}{2}bc\sin A,$$
$$K = \tfrac{1}{2}b^2\tan A = \tfrac{1}{2}a^2\tan B,$$
$$K = \tfrac{1}{2}c^2\sin A\cos A = \tfrac{1}{2}c^2\sin B\cos B.$$

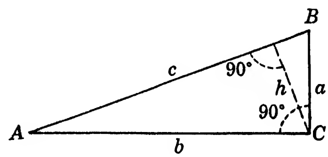


FIG. 1-32.

24. Find the length of line segment y in Fig. 1-33 ($\sin 40° = 0.643$, $\cos 40° = 0.766$, $\tan 40° = 0.839$).

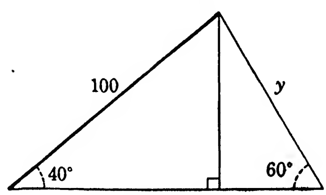


FIG. 1-33.

25. Find length BD in Fig. 1-34 ($\sin 20° = 0.342$, $\cos 20° = 0.940$, $\tan 20° = 0.364$).

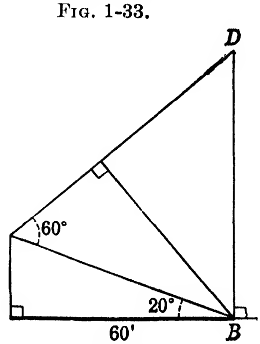


FIG. 1-34.

26. The relative positions of the point A at the bow of an aircraft carrier 660 ft. long, C at its stern, and B on a near-by submarine are shown in Fig. 1-35. If the tangent of angle ABC is $\frac{1}{25}$ and angle CAB is 90°, about how far is the submarine from the carrier?

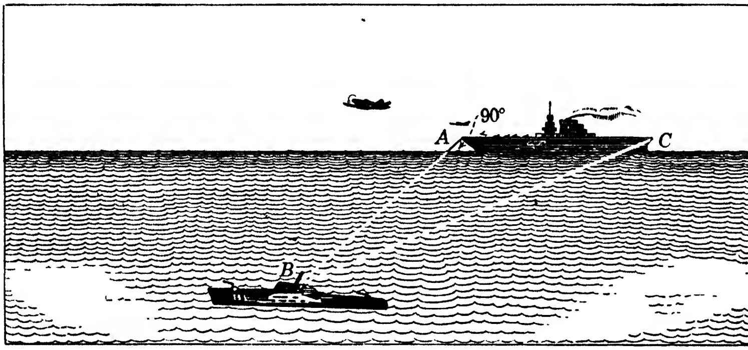


Fig. 1-35.

27. The navigator of a vessel steaming in a given direction observes a light 4 miles distant. If the angle at the vessel between the line of sight to the light and the line containing the ship's keel is 30°, find the vessel's nearest approach to the light.

CHAPTER 2

THE RIGHT TRIANGLE

2-1. The table of values of trigonometric functions. The table on page 20 contains approximate values, accurate to three figures, of the trigonometric functions of angles by degrees from 0 to 90°. The value of a desired function of an angle between 0 and 45° is found in the column headed by the name of the function and in the row having as its first entry the number of degrees in the angle. For example, in the column headed "tan" and in the row having 25° as its first entry, read 0.466. Hence, tan 25° = 0.466. Likewise, cos 37° = 0.799 and sin 18° = 0.309.

If the angle is between 45 and 90°, the value of the function is found in the row with the number of degrees in the angle and in the column at the bottom of which is the name of the function. Thus, sin 65° = 0.906, cos 83° = 0.122 and tan 49° = 1.150.

EXERCISES 2-1

1. Using the table of trigonometric functions, verify each of the following:

(*a*) sin 35° = 0.574. (*b*) cos 70° = 0.342.
(*c*) tan 40° = 0.839. (*d*) sec 17° = 1.046.
(*e*) csc 73° = 1.046. (*f*) cot 65° = 0.466.
(*g*) sin 41° = 0.656. (*h*) cos 88° = 0.035.

2. Find each of the following:

(*a*) tan 12°. (*b*) csc 53°. (*c*) cot 78°.
(*d*) sin 52°. (*e*) cos 9°. (*f*) cos 61°.
(*g*) sec 33°. (*h*) sin 55°. (*i*) tan 24°.

3. Compute, accurate to three decimal places, sin 45°, tan 45°, sin 30°, sec 30°, csc 30°, sin 60°, sec 45°, and compare with the values of these functions found from the table.

4. Find the number of degrees in the angle in each of the following:

(*a*) sin A = 0.407. (*b*) cos B = 0.839. (*c*) tan A = 0.268.
(*d*) cos B = 0.988. (*e*) tan D = 1.881. (*f*) sin E = 0.927.
(*g*) sec A = 1.346. (*h*) csc B = 2.790. (*i*) sec A = 1.701.

Numerical Values of the Trigonometric Functions

Degrees	sin	csc	tan	cot	cos	sec	
0	0.000	∞	0.000	∞	1.000	1.000	**90**
1	0.017	57.299	0.017	57.290	1.000	1.000	89
2	0.035	28.654	0.035	28.636	0.999	1.001	88
3	0.052	19.107	0.052	19.081	0.999	1.001	87
4	0.070	14.336	0.070	14.301	0.998	1.002	86
5	0.087	11.474	0.087	11.430	0.996	1.004	85
6	0.105	9.567	0.105	9.514	0.995	1.006	84
7	0.122	8.206	0.123	8.144	0.993	1.008	83
8	0.139	7.185	0.141	7.115	0.990	1.010	82
9	0.156	6.392	0.158	6.314	0.988	1.012	81
10	0.174	5.759	0.176	5.671	0.985	1.015	**80**
11	0.191	5.241	0.194	5.145	0.982	1.019	79
12	0.208	4.810	0.213	4.705	0.978	1.022	78
13	0.225	4.445	0.231	4.331	0.974	1.026	77
14	0.242	4.134	0.249	4.011	0.970	1.031	76
15	0.259	3.864	0.268	3.732	0.966	1.035	75
16	0.276	3.628	0.287	3.487	0.961	1.040	74
17	0.292	3.420	0.306	3.271	0.956	1.046	73
18	0.309	3.236	0.325	3.078	0.951	1.051	72
19	0.326	3 072	0.344	2.904	0.946	1.058	71
20	0.342	2.924	0.364	2.747	0.940	1.064	**70**
21	0.358	2.790	0.384	2.605	0.934	1.071	69
22	0.375	2.669	0.404	2.475	0.927	1.079	68
23	0.391	2.559	0.424	2.356	0.921	1.086	67
24	0.407	2.459	0.445	2.246	0.914	1.095	66
25	0.423	2.366	0.466	2.145	0.906	1.103	65
26	0.438	2.281	0.488	2.050	0.899	1.113	64
27	0.454	2.203	0.510	1.963	0.891	1.122	63
28	0.469	2.130	0.532	1.881	0.883	1.133	62
29	0.485	2.063	0.554	1.804	0.875	1.143	61
30	0.500	2.000	0.577	1.732	0.866	1.155	**60**
31	0.515	1.942	0.601	1.664	0.857	1.167	59
32	0.530	1.887	0.625	1.600	0.848	1.179	58
33	0.545	1.836	0.649	1.540	0.839	1.192	57
34	0.559	1.788	0.675	1.483	0.829	1.206	56
35	0.574	1.743	0.700	1.428	0.819	1.221	55
36	0.588	1.701	0.727	1.376	0.809	1.236	54
37	0.602	1.662	0.754	1.327	0.799	1.252	53
38	0.616	1.624	0.781	1.280	0.788	1.269	52
39	0.629	1.589	0.810	1.235	0.777	1.287	51
40	0.643	1.556	0.839	1.192	0.766	1.305	**50**
41	0.656	1.524	0.869	1.150	0.755	1.325	49
42	0.669	1.494	0.900	1.111	0.743	1.346	48
43	0.682	1.466	0.933	1.072	0.731	1.367	47
44	0.695	1.440	0.966	1.036	0.719	1.390	46
45	0.707	1.414	1.000	1.000	0.707	1.414	45
	cos	**sec**	**cot**	**tan**	**sin**	**csc**	**Degrees**

2-2. Finding heights and distances by means of trigonometric functions. The following example will illustrate the method to be used.

Example. An angle of a right triangle is 55°, and the adjacent leg is 58 units. Find the remaining parts.

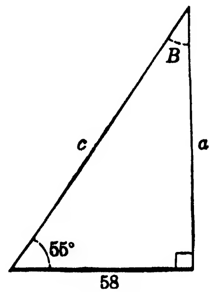


FIG. 2-1.

Solution. In Fig. 2-1,

(*a*) $B = 90° - 55° = \mathbf{35°}$.

(*b*) $\dfrac{a}{58} = \tan 55°$.

From the table, $\tan 55° = 1.428$.

$$\therefore \frac{a}{58} = 1.428.$$

$$\therefore a = 58(1.428), \text{ or } \mathbf{82.8}.$$

(*c*) $\dfrac{c}{58} = \sec 55°$. From the table, $\sec 55° = 1.743$.

$$\therefore \frac{c}{58} = 1.743.$$

$$\therefore c = 58(1.743), \text{ or } \mathbf{101.1}.$$

The following rule may be helpful:

Rule. To find an unknown side of a right triangle when a side and an acute angle are given:

(*a*) Draw a figure on which are written the values of the known parts and a letter for the unknown side.

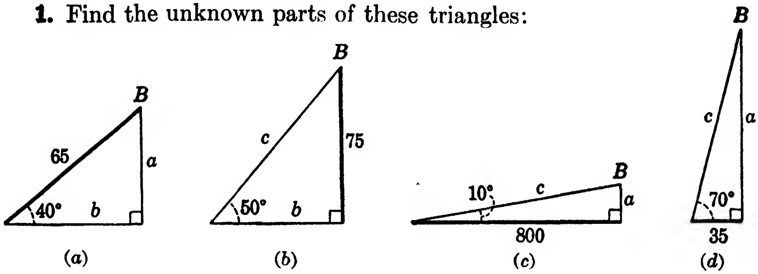
(*b*) Write a formula relating the unknown part with the two known parts.

(*c*) Substitute for the trigonometric function of the angle the value from the table.

(*d*) Solve the resulting equation.

EXERCISES 2-2

1. Find the unknown parts of these triangles:



2. Solve each of the following right triangles, in which the known parts are:

(*a*) $c = 85$, $A = 35°$.	(*b*) $a = 200$, $B = 80°$.	(*c*) $a = 500$, $A = 55°$.
(*d*) $B = 75°$, $c = 20$.	(*e*) $c = 100$, $A = 25°$.	(*f*) $b = 60$, $B = 70°$.

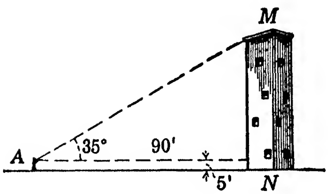


FIG. 2-2.

3. A surveyor wishing to find the height of a tower, represented by MN in Fig. 2-2, stands 90 ft. from its base, measures the angle A, and finds it to be 35°. If the surveyor's eye is 5 ft. above the ground, find the height of the tower.

4. A city block is in the form of a right triangle with a hypotenuse of 300 ft. If one angle is 35°, find the lengths of the other two sides.

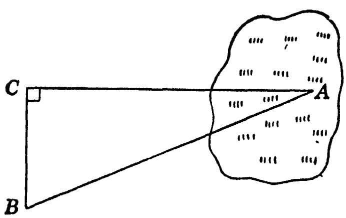


FIG. 2-3.

5. In order to find the distance from C to an inaccessible point A (see Fig. 2-3), line CB, 100 ft. long, was laid off perpendicular to CA, and angle CBA was found to be 70°. Find the distance CA.

6. At a point 55 ft. from the base of a flagpole that is standing on level ground the angle of elevation of the top of the pole is 50°. Find the height of the flagpole, correct to the nearest foot.

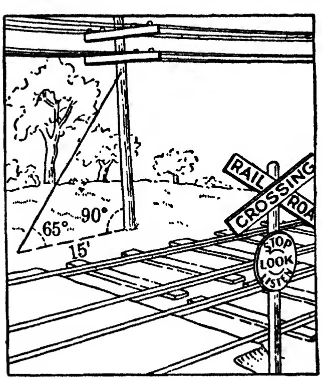


FIG. 2-4.

7. A guy wire from a point 5 ft. from the top of a telephone pole makes an angle of 65° with the level ground and is anchored 15 ft. from the base of the pole, as shown in Fig. 2-4. How high is the pole?

8. An airplane starts from a station and rises at an angle of 10° with the horizontal. By how many feet will it clear a vertical wall 100 ft. high and 900 ft. from the station?

9. An observer in a captive balloon is 985 yd. above level ground. The line of direction of the enemy's outpost makes an angle of 80° with the vertical. How far away is the outpost?

10. When the direction of the sun makes an angle of 35° with the horizontal, an oil derrick casts a shadow 150 ft. long. How high is the derrick (see Fig. 2-5)?

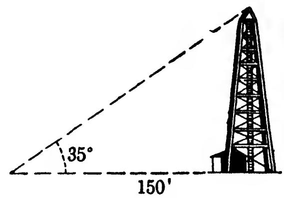


FIG. 2-5.

11. In a certain quartz crystal two of the plane faces of the crystal meet at an angle of 50°. If the perpendicular distance from a point A in one face to the other face is 3 cm., find the distance of A from the intersection of the two faces.

12. A plot of ground is in the form of a right triangle, with one leg 10 yd. long and its adjacent angle 20°. Find the length of a fence surrounding the plot.

13. An observer in the airplane shown in Fig. 2-6 measures the angle ABC and finds it to be 35°. He reads from his altimeter the altitude BC to be 3467 ft. What is the width AC of the island?

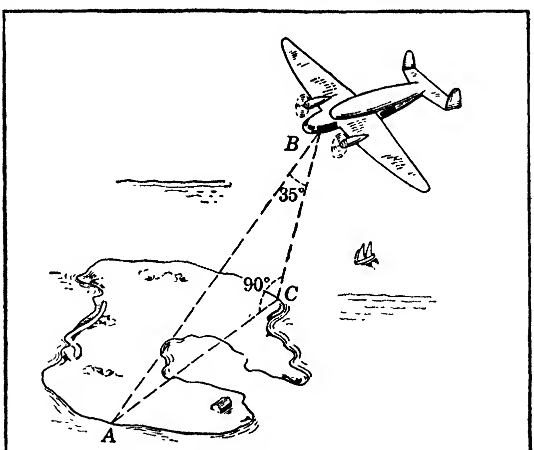


FIG. 2-6.

14. The shortest side of a field in the form of a right triangle is 300 ft. long. If the angle opposite this side is 40°, find the area of the field.

15. At a point A in a horizontal plane through the base of a flagpole the angle of elevation of its top is 35°. If the flagpole is 40 ft. high, find the distance from A to the pole.

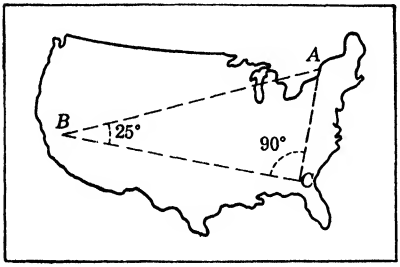


Fig. 2-7.

16. If the map distance BC is 2.5 cm. (see Fig. 2-7) and if angle $ABC = 25°$, find the map distance AB.

17. At a point midway between two trees on a horizontal plane the angles of elevation of their tips are 30° and 60°, respectively. Show that one tree is three times as high as the other.

18. An observer in an airplane (see Fig. 2-8) 2000 ft. above the sea sights two ships A and B and finds their angles of depression to be 44° and 32°, respectively. If the observer is in the same vertical plane with the ships, find the distance AB.

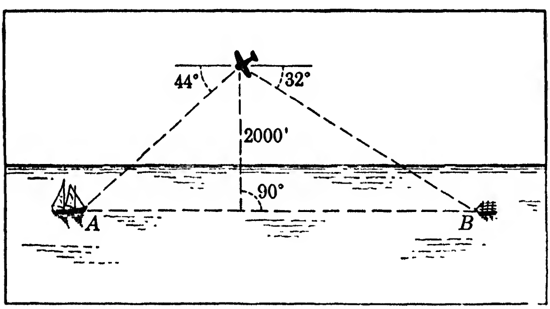


Fig. 2-8.

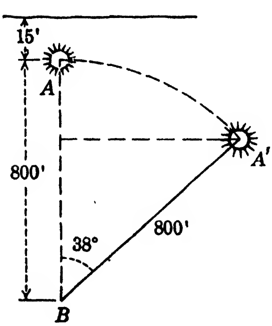


Fig. 2-9.

19. The mine A in Fig. 2-9 is attached to the fixed point B by means of the 800-ft. cable AB. When the cable is vertical, the mine is 15 ft. below the surface of the water. How far from the surface is it when the tidal current has swung it to the position A'?

20. The ship represented in Fig. 2-10 steams at a uniform speed due east. At 7 A.M. its captain observes a lighthouse 10 miles away bearing due north, and at 7:30 A.M. he finds that it bears 40° west of north. Find the speed.

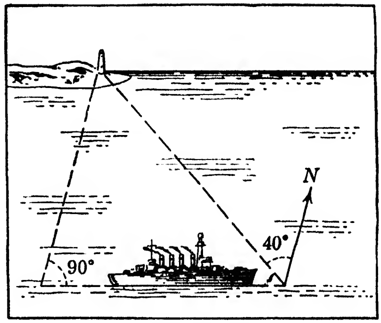


FIG. 2-10.

21. Find all unknown lengths of line segments in Fig. 2-11.

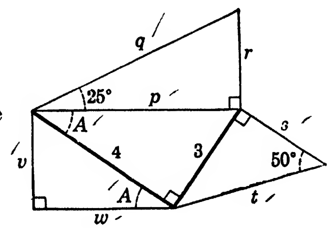


FIG. 2-11.

2-3. Accuracy. Suppose a man knows that his house is longer than 31.5 ft. but shorter than 32.5 ft. How can he express the length of his house on the basis of this meager knowledge? If he should tell an engineer that his house was 32 ft. long, the engineer would be justified in thinking that the length was correct to the nearest foot. Hence he might argue as follows: The house is more than 31.5 ft. long; otherwise 31 ft. would be a closer approximation than 32 ft. Also, the house is shorter than 32.5 ft.; otherwise 33 ft. would be a better approximation. Similarly, if a man gave 32.3 ft. as the length of his house, an engineer would conclude that it was longer than 32.25 ft. but shorter than 32.35 ft. Evidently the error in this case would not be greater than $\frac{5}{100}$ $(= \frac{1}{20})$ ft., or 0.6 in. The first length, 32 ft., would be spoken of as accurate to two significant figures, the second length, 32.3 ft., as accurate to three significant figures. A number is rounded off (or is accurate) to k significant figures when it is expressed, as nearly as possible, by means of a first digit different from zero, $k - 1$ digits immediately following the first, and enough zeros to place the decimal point. Thus

0.000512 ft., 318000 in., 0.308 mile, all represent data accurate to three significant figures. Note that neither the four zeros in 0.000512 nor the three zeros in 318000 are significant, since they serve merely to place the decimal point. The numbers 27862, 0.3996, and 38.85 when rounded off to three figures would be 27900, 0.400, 38.8, respectively. 38.85 might have been rounded off to 38.9; we chose 38.8 because many computers take the even digit when there is a choice.

Results obtained with a 10-in. slide rule are generally considered accurate to three significant figures, although one cannot always be sure of the last figure. With data accurate to four figures four-place logarithm tables are used, with data accurate to five figures, five-place tables are used, etc. The result of computing $0.0038761 \sqrt{4.8724}$ would be written 0.00856 if computed with a 10-in. slide rule, 0.008556 if computed with a four-place logarithm table, and 0.0085560 if computed with a five-place table or a more accurate one.

EXERCISES 2-3

1. Round off each of the following numbers to three figures:

(*a*) 6.7245. (*b*) 984.55. (*c*) 69349. (*d*) 4935.

2. A careless engineer gave the height of a flagpole as 48.672 ft. However, the measurements were made so poorly that his result might have been 2 in. in error. What height should he have given?

2-4. The functions of an angle in degrees and minutes. If the angle is not an exact number of degrees, the value of a function of the angle may be found by interpolation. For example, to find sin 57°24′, take from the table the values of sin 57° and sin 58°, and make the following form:

$$60' \left\{ \begin{array}{l} 24' \left\{ \begin{array}{l} \sin 57°00'' = 0.839 \\ \sin 57°24' \;\;= \;? \end{array} \right\} d \\ \qquad \sin 58°00'' = 0.848 \end{array} \right\} 9.$$

For small changes in an angle, the increment of angle is nearly proportional to the increment of its sine. Therefore

$$\tfrac{24}{60} = \frac{d}{9} \text{ (nearly)}, \qquad \text{or} \qquad d = (\tfrac{24}{60})(9) = 4 \text{ (nearly)}.$$

Adding 0.004 to 0.839, we obtain

$$\sin 57°24' = \mathbf{0.843}.$$

When the value of the function is given, a similar process enables us to find the angle. For example, if $\tan \theta = 0.734$, to find θ we use the table to get $\tan 36° = 0.727$, $\tan 37° = 0.754$, and then make the following form:

$$60' \left\{ \begin{array}{l} x' \left\{ \begin{array}{l} \tan 36° = 0.727 \\ \tan \quad \theta = 0.734 \end{array} \right\} 7 \\ \quad\; \tan 37° = 0.754 \end{array} \right\} 27.$$

As before, we write $\dfrac{x'}{60} = \frac{7}{27}$, or $x' = (\frac{7}{27})(60') = 16'$ (nearly). Therefore, $x = \mathbf{36°16'}$.

EXERCISES 2-4

Find the value of each of the following:

1. $\sin 42°40'$. **2.** $\cos 54°23'$. **3.** $\tan 22°10'$.
4. $\cot 20°35'$. **5.** $\sec 62°20'$. **6.** $\csc 16°18'$.
7. $\sin 12°4'$. **8.** $\cos 15°11'$. **9.** $\tan 63°29'$.
10. $\cos 45°34'$. **11.** $\cot 73°54'$. **12.** $\sin 57°42'$.

For each of the following equations, find an acute angle satisfying it:

13. $\sin \theta = 0.672$. **14.** $\cos \theta = 0.908$. **15.** $\tan \theta = 1.630$.
16. $\cot \theta = 0.518$. **17.** $\sec \theta = 1.200$. **18.** $\csc \theta = 3.256$.
19. $\sin \theta = 0.841$. **20.** $\cos \theta = 0.723$. **21.** $\tan \theta = 0.482$.

Solve the following right triangles:

22. $a = 32$, $A = 48°25'$. **23.** $c = 46.1$, $B = 29°14'$.
24. $a = 16.3$, $c = 25.1$. **25.** $a = 3.04$, $b = 2.51$.
26. $b = 67$, $B = 32°15'$. **27.** $c = 47.6$, $A = 62°12'$.
28. $a = 41$, $b = 20$. **29.** $c = 37$, $A = 69°50'$.

2-5. Definitions. The terms defined below will be used in the following list of problems and elsewhere in this book.

The **line of sight** is a straight line connecting the eye of an observer with the object viewed.

The **angle of elevation** at a point O of an observed point B higher than O is the angle that the straight line OB makes with the horizontal.

The **angle of depression** at a point C of an observed point O lower than C is the angle that the straight line CO makes with the horizontal.

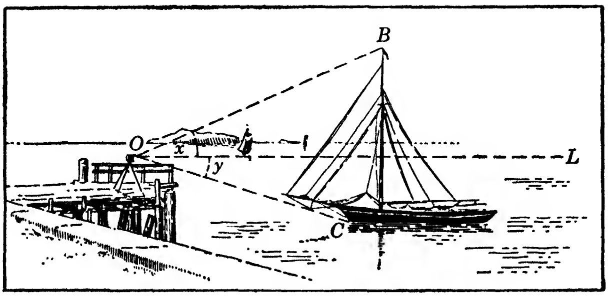


FIG. 2-12.

The **angle subtended by a line** BC at a point O is the angle formed by the rays OB and OC.

For example, in the vertical plane OBC represented in Fig. 2-12, OB is the line of sight for an observer at O viewing the point B, angle x is the angle of elevation of B at O, angle y is the angle of depression of C at O, and angle BOC is the angle subtended at O by the line BC.

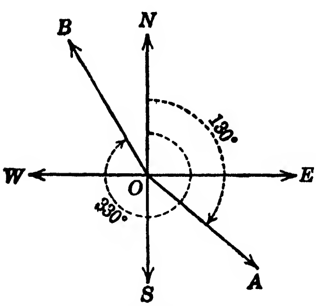


FIG. 2-13.

The **compass bearing** of an object is the angle, measured clockwise, that is, from north around toward or through east, between a horizontal line running north from an observer and a horizontal line connecting the observer with the object. The angle measured clockwise in a horizontal plane from north to the direction of motion of an observer is known as his **compass course.**

Thus the bearing of point A for an observer at O in Fig. 2-13 is 130°; the bearing of B is 330°. A ship sailing from O toward A would have a compass course of 130°. The direction to an object is often indicated by stating an initial direction, north ($N.$) or south ($S.$), then the angle in degrees, minutes, and seconds, and

finally a letter indicating whether the object is east (*E.*) or west (*W.*) of the observer. Thus the bearing of A in Fig. 2-13 might be given as S. 50° E. and that of B as N. 30° W.

EXERCISES 2-5

1. The master of a whaling vessel orders his mate to take a position 500 yd. from his ship in a small boat, as shown in Fig. 2-14. The top of

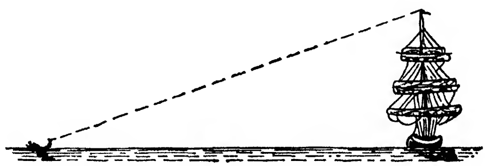


FIG. 2-14.

the whaling vessel's mast above the water line is 213 ft. Find what angle this height will subtend on the mate's sextant when he reaches his position.

2. A ship moving due west at 15 miles per hour passes due north of a given point A, and 20 min. later it bears N. 38°26′ W. from the given point. Find the distance of the ship from A at both times.

3. A surveyor in a barn distant 1 mile from a railroad track observes that a train of cars on the track subtends 35°40′ at his position when one

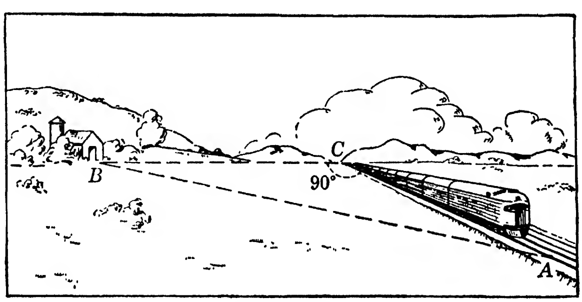


FIG. 2-15.

end of the train is directly opposite him. How long is the train (see Fig. 2-15)?

4. From the top of a rock that rises vertically 80 ft. out of the water the angle of depression of a boat is found to be 35°. Find the distance of the boat from the foot of the rock.

5. The shadow of a vertical cliff 113 ft. high just reaches a boat on the sea 93 ft. from its base. Find the altitude of the sun.

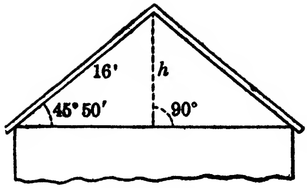


FIG. 2-16.

6. The rafters of a house make an angle of 45°50′ with the horizontal and are 16 ft. long from the top of the wall to the highest point of the roof. Find the height h of the roof above the wall (see Fig. 2-16).

7. The two stations A and B shown in Fig. 2-17 are 5200 ft. apart. When an airplane D was directly above A, an observer at B found the

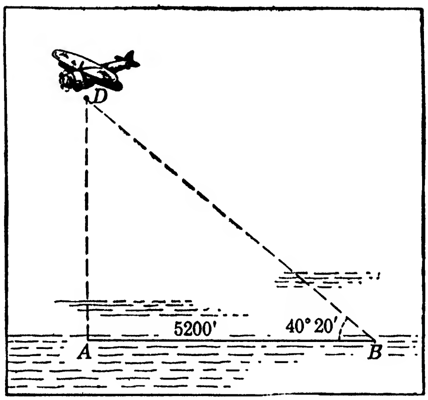


FIG. 2-17.

angle of elevation of the plane to be 40°20′. Find the distance from the plane to station B.

8. From a point 1420 ft. above a trench, an observer in an airplane finds that the angle of depression of an enemy fort is 23°50′. How far is the trench from the fort?

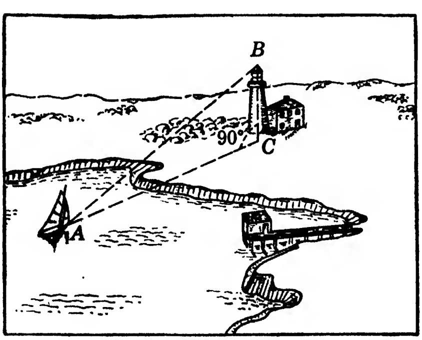


FIG. 2-18.

9. From a point A, 175 ft. from the base of a lighthouse, a yachtsman finds the angle of elevation of the top to be 29°30′ (see Fig. 2-18). Find the height of the lighthouse.

10. From an observer's position O, 8.5 ft. above the water (see Fig. 2-19), the angle of elevation of the top B of the sail was found to be

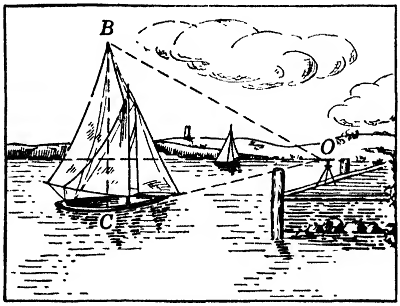


FIG. 2-19.

28°30′, and the angle of depression of the lowest point C was 20°25′. Find the total height BC of the sailboat.

11. From the top of a hill the angles of depression of two successive milestones on a straight level road leading to the hill are observed to be 5° and 15°. How high is the hill?

2-6. Solution of the right triangle by slide rule.* A fundamental law of trigonometry, called *the law of sines*, is especially adapted to slide-rule computation. It states that the ratio of the sine of any angle of a triangle to the opposite side is equal to the ratio of the sine of any second angle to its opposite side; or, in symbols,

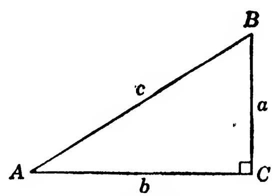


FIG. 2-20.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \qquad (1)$$

To prove this for a right triangle, use Fig. 2-20 to obtain

$$\frac{a}{c} = \sin A, \qquad \text{or} \qquad \frac{1}{c} = \frac{\sin A}{a}, \qquad (2)$$

$$\frac{b}{c} = \sin B, \qquad \text{or} \qquad \frac{1}{c} = \frac{\sin B}{b}. \qquad (3)$$

Equating the values of $1/c$ in (2) and (3), we get

$$(\sin A)/a = (\sin B)/b = 1/c,$$

* A good preparation for making the computations of this article and the next one may be obtained by studying Arts. 14-17, 14-18.

or replacing 1 by its equal, sin 90°,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin 90°}{c}. \tag{4}$$

To solve the triangle of Fig. 2-21, substitute 35° for A, 387 for a, and 55° for B in (4) to obtain

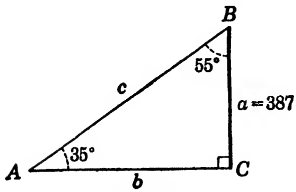


FIG. 2-21.

$$\frac{S}{D}: \frac{\sin 35°}{387} = \frac{\sin 55°}{b} = \frac{\sin 90°}{c}, \tag{5}$$

where the symbol S/D indicates that the angles in the numerator are to be set on the S scale of the slide rule, and the denominators on the D scale. Hence, in accordance with the proportion principle (Fig. 2-22),

push hairline to 387 on D,
draw 35° of S under the hairline,
push hairline to 55° on S,
at the hairline read b = **552** on D;
push hairline to 90° on S,
at hairline read c = **675** on D.

The student should note that it is unnecessary to write the law of sines to solve a right triangle. Observing that, in accord-

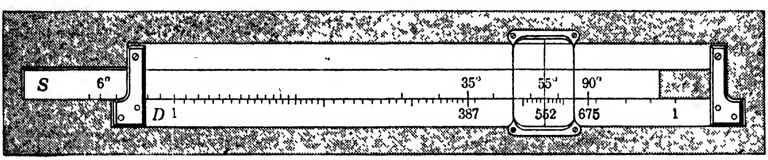


FIG. 2-22.

ance with the law of sines, each side and the angle opposite must be set opposite each other on the slide rule, he uses the following rule:

Rule. To solve a right triangle, except when the given parts are two legs, draw the triangle and write on each known part, including the 90° angle, its value, and then

push the hairline to known side on *D*,
draw angle opposite on *S* under hairline,

push hairline to any other known side on D;
at the hairline read angle opposite on S,
push hairline to any known angle on S,
at the hairline read side opposite on D.

EXERCISES 2-6

Solve the following right triangles by means of the slide rule.

1. $a = 60$, $c = 100$.	**2.** $b = 200$, $A = 64°$.	**3.** $b = 47.7$, $B = 62°56'$.
4. $a = 50.6$, $A = 38°40'$.	**5.** $c = 37.2$, $B = 6°12'$.	**6.** $a = 0.624$, $c = 0.910$.
7. $a = 729$, $B = 68°50'$.	**8.** $c = 11.2$, $A = 43°30'$.	**9.** $a = 83.4$, $A = 72°7'$.

2-7. Slide-rule solution of a right triangle when two legs are known. When the two legs of a right triangle are known, the smaller acute angle may be found from its tangent, the other acute angle by subtracting the smaller one from 90°, and then the hypotenuse by using the law of sines. Thus, to solve the right triangle shown in Fig. 2-23, write

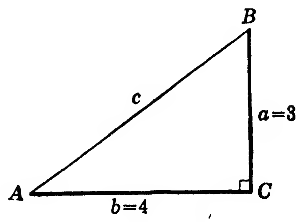


FIG. 2-23.

$$\tan A = \frac{3}{4} \qquad \text{or} \qquad \frac{\tan A}{3} = \frac{1}{4}$$

Hence, in accordance with the proportion principle [Fig. 2-24(a)],

set the index of T to 4 on D,
push hairline to 3 on D,
at the hairline read A = **36°52′** on T.

Evidently angle $B = 90° - A =$ **53°8′**. To find the hypotenuse c, apply the setting based on the law of sines explained in Art. 2-6 [see Fig. 2-24(b)].

push hairline to 3 on D,
draw 36°52′ on S under the hairline,
at the index of S read c = **5** on D.

If one observes that the first of the three steps just indicated is unnecessary, since the hairline was already set to 3 on D when the angle A was found, he will see that the following rule applies:

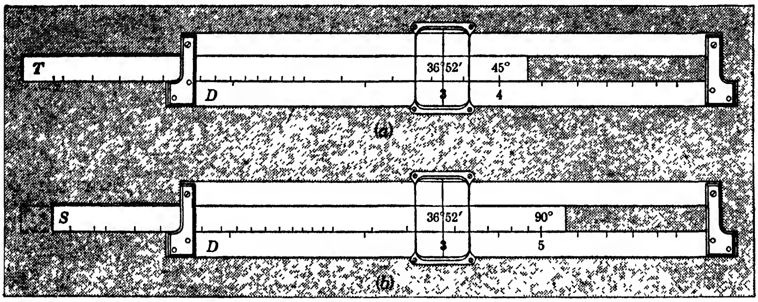


FIG. 2-24.

Rule. To solve a right triangle when two legs are known:

> **To greater leg on D set proper index of slide,**
> **push hairline to smaller leg on D,**
> **at the hairline read smaller acute angle on T,**
> **draw this angle on S under the hairline,**
> **at index of slide read hypotenuse on D.**

EXERCISES 2-7

Solve the following right triangles by means of the slide rule:

1. $a = 12.3$, $b = 20.2$. **2.** $a = 273$, $b = 418$. **3.** $a = 13.2$, $b = 13.2$.
4. $a = 101$, $b = 116$. **5.** $a = 28$, $b = 34$. **6.** $a = 42$, $b = 71$.
7. $a = 50$, $b = 23.3$. **8.** $a = 12$, $b = 5$. **9.** $a = 0.31$, $b = 4.8$.

2-8. Table of logarithms of trigonometric functions. When a high degree of accuracy is desired for the solution of a problem involving trigonometry, the computation should be made by means of logarithms. To facilitate the process, tables of logarithms of the trigonometric functions have been prepared. The sample page printed in the next article is a page from such a table. The complete table* gives, accurate to four decimal

* See Table II, pp. 327 to 331.

Trigonometric Functions

Angles	Sines Nat.	Sines Log.	Cosines Nat.	Cosines Log.	Tangents Nat.	Tangents Log.	Cotangents Nat.	Cotangents Log.	Angles
27° 00	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	.6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
Angles	Cosines Nat.	Cosines Log.	Sines Nat.	Sines Log.	Cotangents Nat.	Cotangents Log.	Tangents Nat.	Tangents Log.	Angles

places, the logarithms of four trigonometric functions for angles from 0 to 90° at intervals of 10 min. Note that the table contains the natural functions as well as the logarithms of the functions.

2-9. To find the logarithms of a trigonometric function of an angle. The solution of the following examples illustrates the method of finding the logarithms of a trigonometric function of a given angle.

Example 1. Find log sin 29°42′.

Solution. From the table we find the logarithms in the following form and then compute the differences shown.

$$\left.\begin{array}{l}\left.\begin{array}{l}\log \sin 29^\circ 40'\\ \log \sin 29^\circ 42'\end{array}\right\}2'\\ \log \sin 29^\circ 50'\end{array}\right\}10' \quad \left.\begin{array}{l}\left.\begin{array}{l}= 9.6946 - 10\\ = x\end{array}\right\}y\\ = 9.6968 = 10\end{array}\right\}d = 0.0022.$$

The small changes in angle are nearly proportional to the corresponding changes in logarithms. Therefore,

$$\frac{y}{0.0022} = \frac{2}{10}, \qquad \text{or} \qquad y = \frac{2}{10}(0.0022) = 0.0004 \text{ (nearly)},$$

and log sin 29°42′ = 9.6946 − 10 + 0.0004 = **9.6950 − 10.**

Example 2. Find log cos 57°16′.

Solution.

$$\left.\begin{array}{l}\left.\begin{array}{l}\log \cos 57^\circ 10'\\ \log \cos 57^\circ 16'\end{array}\right\}6'\\ \log \cos 57^\circ 20'\end{array}\right\}10' \quad \left.\begin{array}{l}\left.\begin{array}{l}= 9.7342 - 10\\ = x\end{array}\right\}y\\ = 9.7322 - 10\end{array}\right\}d = 0.0020.$$

$$\frac{y}{0.0020} = \frac{6}{10} \qquad \text{or} \qquad y = \frac{6}{10}(0.0020) = 0.0012.$$

Since the logarithm of the cosine decreases as the angle increases, log cos 57°16′ = 9.7342 − 10 − 0.0012 = **9.7330 − 10.**

Note that, as the angle increases, the natural sines and tangents and their logarithms increase, while the cosines and cotangents and their logarithms decrease.

EXERCISES 2-8

Find the value of the following:

1. log sin 39°46′.

2. log sin 59°31′.

3. log cos 81°21′.

4. log tan 28°29′.

5. log cot 49°16′.

6. log sin 64°47′.

7. log tan 20°11′.

8. log cos 16°17′.

9. log sin 81°19′.

10. log cos 12°19′.

2-10. To find the angle when the logarithm is given. The solution of the following examples illustrates the method of finding an angle when the logarithm of a trigonometric function of the angle is given.

Example 1. Find the acute angle B when log tan B = 0.1492.

Solution. Observe that 0.1492 lies between the two entries 0.1467 and 0.1494 on the sample page in the column with "Tangents" printed at its foot. Therefore, write the logarithms in the following form and compute the differences as shown:

$$\left.\begin{matrix}\left.\begin{matrix}\log\tan 54^\circ 30' \\ \log\tan B\end{matrix}\right\} y \\ \log\tan 54^\circ 40'\end{matrix}\right\} 10' \quad \left.\begin{matrix}\left.\begin{matrix}= 0.1467 \\ = 0.1492\end{matrix}\right\} 0.0025 \\ = 0.1494\end{matrix}\right\} d = 0.0027.$$

The small changes in angle are nearly proportional to the small changes in logarithm. Therefore,

$$\frac{y}{10} = \frac{0.0025}{0.0027}, \qquad \text{or} \qquad y = \frac{25}{27}(10) = 9',$$

and

$$B = 54^\circ 30' + 9' = \mathbf{54^\circ 39'}.$$

Example 2. Find acute angle B when log cot B = 0.2670.

Solution.

$$\left.\begin{matrix}\left.\begin{matrix}\log\cot 28^\circ 20' \\ \log\cot B\end{matrix}\right\} y \\ \log\cot 28^\circ 30'\end{matrix}\right\} 10' \quad \left.\begin{matrix}\left.\begin{matrix}= 0.2683 \\ = 0.2670\end{matrix}\right\} 0.0013 \\ = 0.2652\end{matrix}\right\} 0.0031.$$

$$\frac{y}{10} = \frac{0.0013}{0.0031}, \qquad \text{or} \qquad y = \frac{0.0013}{0.0031}(10) = 4',$$

and

$$B = 28°20' + 4' = \mathbf{28°24'}.$$

EXERCISES 2-9

Find the value of A in the following:

1. $\log \sin A = 9.3146 - 10.$
2. $\log \tan A = 9.0314 - 10.$
3. $\log \cot A = 0.0121.$
4. $\log \sin A = 9.1286 - 10.$
5. $\log \cos A = 9.9214 - 10.$
6. $\log \cos A = 9.2161 - 10.$
7. $\log \tan A = 0.1116.$
8. $\log \cot A = 9.8619 - 10.$
9. $\log \sin A = 9.0221 - 10.$
10. $\log \sin A = 8.9578 - 10.$

2-11. Solution of the right triangle by means of logarithms. To solve a right triangle by means of logarithms, proceed as indicated in Art. 2-2, but do the computation with a table of logarithms. The solution of the following example will indicate a very convenient form for the computation, as well as the method of procedure.

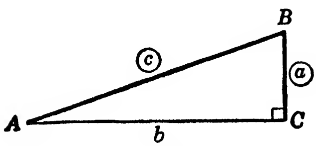


FIG. 2-25.

Example. Solve the right triangle in which $c = 796.5$, $a = 267.5$.

Solution. Fig. 2-25 shows the given parts encircled. From it we obtain $B = 90° - A$. Also,

(a) $\text{Sin } A = \dfrac{a}{c} = \dfrac{267.5}{796.5}.$

$$\begin{aligned} \log 267.5 &= 2.4273 \\ \text{colog } 796.5 &= \underline{7.0988 - 10} \\ \log \sin A &= 9.5261 - 10 \\ \therefore A &= \mathbf{19°37'}. \end{aligned}$$

(b) $\dfrac{b}{c} = \cos A$, or $b = c \cos A = 796.5 \cos 19°37'.$

$$\begin{aligned} \log 796.5 &= 2.9012 \\ \log \cos 19°37' &= \underline{9.9740 - 10} \\ \log b &= 2.8752 \\ \therefore b &= \mathbf{750.2}. \end{aligned}$$

(c) $\dfrac{b}{a} = \cot A$, or $b = a \cot A = 267.5 \cot 19°37'$.

$$\begin{aligned} \log 267.5 &= 2.4273 \\ \log \cot 19°37' &= \underline{0.4481} \\ \log b &= 2.8754 \\ \therefore\ b &= \mathbf{750.2.} \end{aligned}$$

Solution (c) may serve as a check. The same result is obtained as in (b).

The following form is recommended as a compact way of arranging the work. It contains all the numbers used in the computation, including the results. Note that every expression on any line refers to the first number in the line. Note also that l is used to abbreviate the word *log*.

	(a)	(b)	(c)
$a = 267.5$	$\log a = 2.4273$		$\log a = 2.4273$
$c = 796.5$	$\operatorname{colog} c = 7.0988 - 10$	$\log c = 2.9012$	
$A = \mathbf{19°37'}$	$l \sin A = 9.5261 - 10$	$l \cos A = 9.9740 - 10$	$l \cot A = 0.4481$
$b = \mathbf{750.2}$		$\log b = 2.8752$	$\log b = 2.8754$
$B = 90° - A = \mathbf{70°23'}$			

EXERCISES 2-10

Solve the following right triangles:

1. $b = 14$, $A = 35°$.	**2.** $c = 6.275$, $B = 18°47'$.	**3.** $c = 1201$, $a = 885.6$.
4. $a = 8.678$, $b = 2.463$.	**5.** $c = 672.3$, $A = 35°16'$.	**6.** $a = 645.3$, $b = 396.2$.
7. $c = 98.24$, $a = 95.57$.	**8.** $B = 27°9'$, $a = 35.42$.	**9.** $A = 44°10'$, $c = 24.89$.
10. $a = 3.291$, $b = 5.784$.	**11.** $a = 72.13$, $A = 76°17'$.	**12.** $c = 1672$, $B = 83°21'$.

13. A stay wire for a telephone pole is to be attached to the pole 18 ft. 6 in. above the ground and to make an angle of 42°10′ with the horizontal. Find the length of the stay wire, allowing 3 ft. to make attachment.

14. If a ship sails a course of 19° for 201.85 miles, what is the departure?

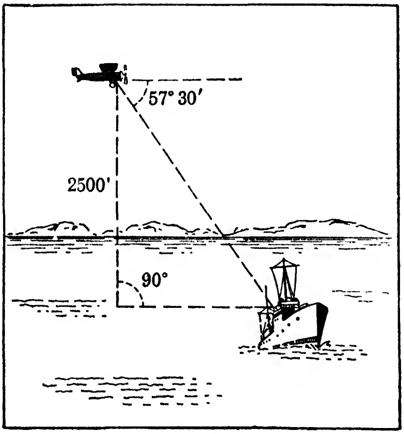


FIG. 2-26.

15. An observer in an airplane 2500 ft. above the sea sights a destroyer at an angle of depression of 57°30′, as shown in Fig. 2-26. Find the distance between the plane and the destroyer.

16. If a railroad track rises 30 ft. 4 in. in a horizontal distance of 5280.7 ft., what is its angle of inclination with the horizontal?

17. The area of a right triangle is 23.58 sq. ft., and one angle is 52°24′. Find the length of the hypotenuse. (See Ex. 23, page 17.)

18. A diagonal of a cube intersects a diagonal of one of its faces. Find the angle between these diagonals.

19. A marble $\frac{3}{4}$ in. in diameter subtends an angle of 2°15.5′ at the eye of an observer. How far is it from the observer?

20. If two straight stretches of railway were extended, they would meet at a point making an angle of 46°18′ with each other. These two stretches are to be connected by means of a circular arc of radius 4500 ft. Find the distance from the point of tangency to the point of intersection of the straight stretches.

21. A rectangular bin is 42 in. long and 30 in. wide. What angles does a vertical, diagonal partition make with the sides of the bin?

22. In building a suspension bridge a straight cable is run from the top of a pier to a point 852 ft. 7 in. from its foot. If from this point the angle of elevation of the top of the pier is 27°6′, what length of cable is required?

23. In a level field a tunnel was dug into the earth at an angle of 19°20′ with the horizontal. At a point in the field 285 ft. from the entrance of the tunnel an engineer dug a vertical shaft to meet the tunnel. Find the depth of this shaft.

24. Assuming that the earth is a sphere of radius 3958.5 miles, how far is a point in latitude 41°40′ from the earth's axis?

25. On a 2 per cent railroad grade, that is, a rise of 2 ft. in each 100 ft. measured horizontally, what is the angle at which the rails are inclined to the horizontal? How far must one move along the rails to be 162 ft. higher than at the starting point?

26. Find the radii of the inscribed and circumscribed circles of a regular octagon whose side is 6.254.

27. At a point A due west of the Washington Monument, which is 555 ft. high, the angle of elevation of its top was observed to be $51°22.9'$. Find the angle of elevation of the monument at another point B, 200 ft. west of A, assuming that the points A and B and the base of the monument are in the same horizontal plane.

2-12. Solution of rectilinear figures. An expression is convenient for logarithmic computation if its evaluation involves only multiplications and divisions. To obtain such an expression for an unknown length in a rectilinear figure, one generally drops perpendiculars in such a way as to form a chain of right triangles, each of which has a side in common with the next one in the chain. The first triangle has a side of known length, and the last one has as a side the length to be found. The following example will illustrate the procedure.

Example. A surveyor on a mountain peak observes below him two ships lying at anchor 1 mile apart and in the same vertical plane with his position. He finds the angles of depression of the ships to be 18° and 10°, respectively. How high does the peak rise above the water?

Solution. In Fig. 2-27, H represents the position of the surveyor, M and N represent the respective positions of the ships,

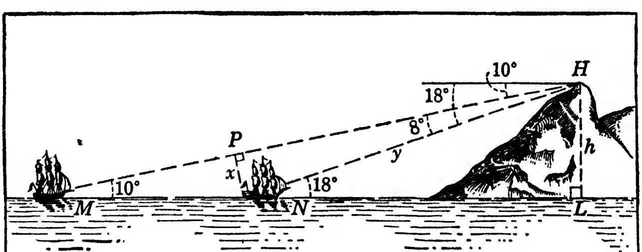


FIG. 2-27.

and the angles marked 10° and 18° represent the angles of depression. Draw NP perpendicular to MH, and denote the

length of NP by x and that of NH by y. From triangle MNP,

$$\frac{x}{5280} = \sin 10°, \qquad \text{or} \qquad x = 5280 \sin 10°. \qquad (a)$$

From triangle NPH,

$$\frac{y}{x} = \csc 8°, \qquad \text{or} \qquad y = x \csc 8°. \qquad (b)$$

From triangle LNH,

$$\frac{h}{y} = \sin 18°, \qquad \text{or} \qquad h = y \sin 18°. \qquad (c)$$

Substituting the value of y from (b) and x from (a) in (c), we obtain

$$h = y \sin 18° = x \csc 8° \sin 18° = 5280 \sin 10° \csc 8° \sin 18°.$$

The following form shows the computation:

$$\begin{aligned}
\log 5280 &= 3.7226 \\
\log \sin 10° &= 9.2397 - 10 \\
\log \csc 8° = \text{colog} \sin 8° &= 0.8564 \\
\log \sin 18° &= 9.4900 - 10 \\
\hline
\log h &= 23.3087 - 20 \text{ or } 3.3087 \\
\therefore h &= \mathbf{2036 \text{ ft.}}
\end{aligned}$$

EXERCISES 2-11

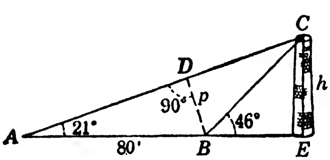


Fig. 2-28.

1. Two points A and B 80 ft. apart lie on the same side of a tower and in a horizontal line through its foot. If the angle of elevation of the top of the tower at A is 21° and at B is 46°, find the height of the tower (see Fig. 2-28).

2. Two points A and B 180 ft. apart lie on the same of side a tower on a hill and in a horizontal line passing directly under the tower. The angles of elevation of the top and bottom of the tower viewed from B are 42° and 34°, respectively, and at A the angle of elevation of the bottom is 10°. Find the height of the tower.

Hint. Draw Fig. 2-29, compute angle $ACB = 24°$, angle $EBC = 8°$, and note that angle $ECF = 42°$. Find in order p_1, BC, p_2, and h.

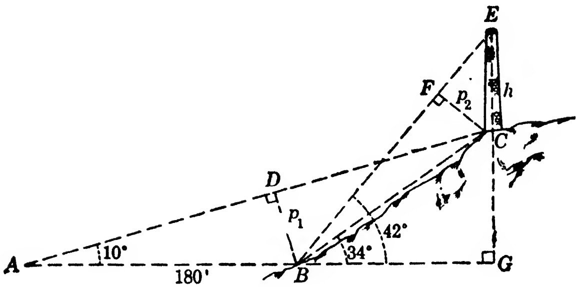


FIG. 2-29.

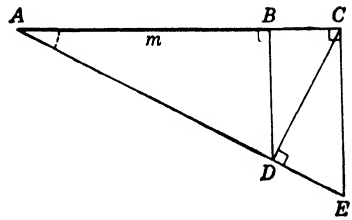


FIG. 2-30.

3. (a) Express BC, DE, and CE in terms of m and A (see Fig. 2-30).

(b) Given $m = 1.96$ in. and $\tan A = 0.482$, find BC, DE, and CE.

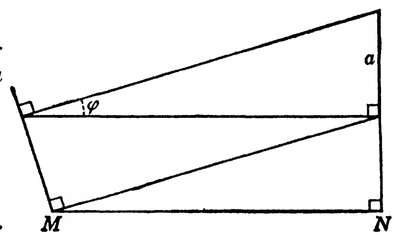


FIG. 2-31.

4. (a) Express all line segments of Fig. 2-31 in terms of a and φ.

(b) Given $a = 34.37$,

$$\tan \varphi = 0.3052,$$

use logs to find the length of MN.

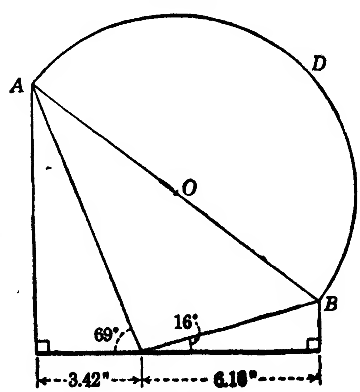


FIG. 2-32.

5. Find the length of diameter AOB, the length of arc ADB, and the area of the semicircle shown in Fig. 2-32.

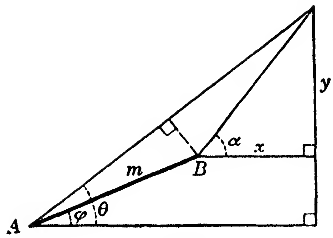


Fig. 2-33.

6. Given the angles α, φ, θ and the distance $AB = m$ in Fig. 2-33, find formulas for x and y.

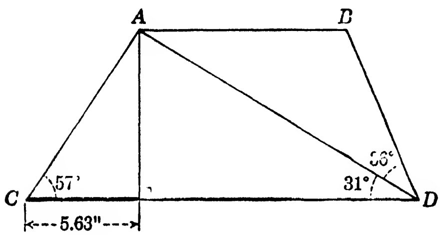


Fig. 2-34.

7. Given AB parallel to CD, in Fig. 2-34, find the area of the figure $ABDC$.

8. A mountain peak C is 4135 ft. above sea level, and from C the angle of elevation of a second peak B is 5°. An aviator at A directly

Fig. 2-35.

over peak C finds that angle CAB is 43°50′ when his altimeter shows that he is 8460 ft. above sea level. Find the height of peak B (see Fig. 2-35).

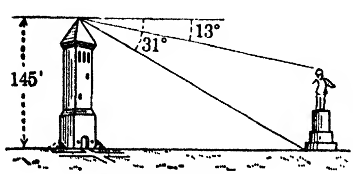


Fig. 2-36.

9. A tower and a monument stand on a level plain (see Fig. 2-36). The angles of depression of the top and bottom of the monument viewed from the top of the tower are 13° and 31°, respectively; the height of the tower is 145 ft. Find the height of the monument.

10. As the altitude of the sun decreased from 63°46′ to 50°35′, the length of the shadow of a tower increased 89.65 ft. Find the height of the tower.

11. Figure 2-37 represents a 600-ft. radio tower. AC and AD are two cables in the same vertical plane anchored at two points C and D on a level with the base of the tower. The angles made by the cables with the horizontal are 44° and 58° as indicated. Find the lengths of the cables and the distance between their anchor points.

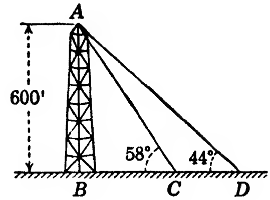


FIG. 2-37.

12. A building and a tower stand on the same horizontal plane, the tower being 120 ft. high. From the top of the tower the angles of depression of the top and bottom of the building are 22°13.8′ and 44°18.9′, respectively. Find the height of the building.

13. A line AB along one bank of a stream is 315 ft. long, and C is a point on the opposite bank. The angle BAC is 66°30′, and the angle ABC is 54°45′. Find the width of the stream.

14. From a ship two lighthouses bear N. 40° E. After the ship sails at 15 knots on a course of 135° for 1 hr. 20 min., the lighthouses bear 10° and 345°.

(*a*) Find the distance between the lighthouses.

(*b*) Find the distance from the ship in the latter position to the farther lighthouse.

2-13. Nautical applications. When a ship sails a comparatively short distance from a point P to a point P' so as to cut at a constant angle α all meridians crossed by it, we use the words **departure (Dep), difference in latitude (DL), distance,** and **course** in speaking of the trip. To understand the meaning of these words, consider the triangular figure $PP'N$ in Fig. 2-38 in which PP' represents the path of a ship, PN represents an arc of a meridian, and NP' represents a small circle, all points of which have the same latitude. For practical purposes we consider this triangle as a plane right triangle and call **NP′** the **departure, PN** the **difference in latitude, PP′** the **distance,** and **angle α** the **course.** The course angle α is measured from the north around through the east from 0° to 360°.

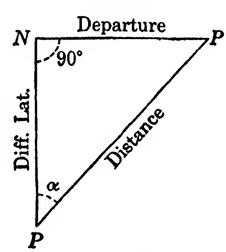


FIG. 2-38.

The words connected with Fig. 2-39 will be used in some of the problems that follow. Observe that the bow of a ship is the forward part, and the stern the rear part. For a man standing on the ship and facing its bow, the side of the ship on his *left* is called the *port side*, and the side on his *right* the *starboard side.* Objects on his left are spoken of as *bearing to port*, and objects on his right as *bearing to starboard.* An object L is said to be

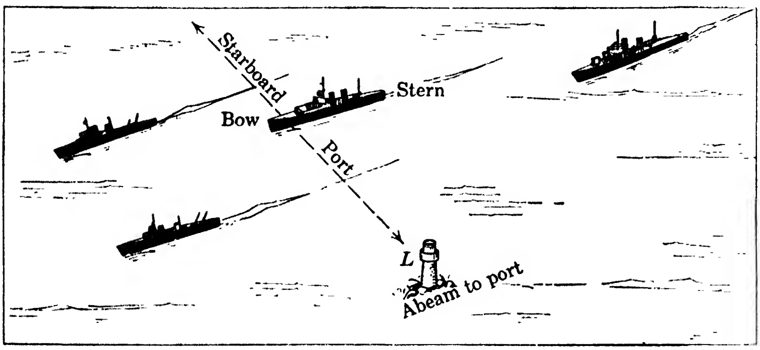


FIG. 2-39.

abeam when the line from L to the pilot of the ship is *perpendicular to the ship's track.* For example, the lighthouse L in Fig. 2-39 is *abeam to port.*

EXERCISES 2-12

1. A ship sails on a course of 42° for 190 miles. Find its departure and difference in latitude.

2. A vessel steamed north 72.4 miles and then east 30.5 miles. Find the course and the distance made good.

3. A ship steams 72.4 miles on a fixed course and its departure is 30.5 miles. Find the course and difference in latitude.

4. Find the departure and the difference in latitude traveled by a ship which sails

(*a*) On a course of 62° for 200 miles,
(*b*) On a course of 143° for 150 miles,
(*c*) On a course of 252° for 300 miles,
(*d*) On a course of 310° for 250 miles.

5. A tanker runs on a course of 321° until its departure is 113 miles. How many miles did she steam?

6. A ship steams at 25 knots for 5 hr. on a course of 75°. Find the departure and the difference in latitude.

7. A vessel heads N 23° W. for 4 hr. at 9.5 knots through a current flowing S. 67° W. at 2.5 knots. Find the true course and the distance made good. In Fig. 2-40, line AC indicates the track.

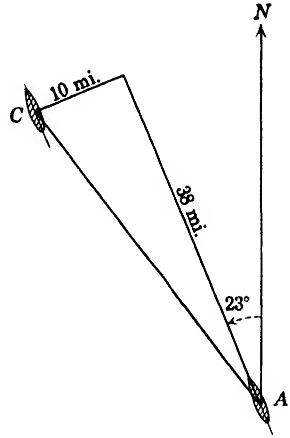


FIG. 2-40.

8. A ship steaming north sights another ship abeam to starboard and distant 4 miles. She steams 48 min. at 10 knots and then the other vessels is still abeam to starboard and distant 1 mile. Find the course and speed of the other vessel.

9. A tower bears 123° from a submarine running on course 169° at 10 knots. Forty-five minutes later the tower is abeam. Find the distance off when abeam. If the tower is 246 ft. high, find the vertical angle subtended by it when abeam.

10. A cruiser steaming on course 78° sights a light bearing 40° from north. If it was abeam after a 5-mile run, find its distance abeam.

11. A point of land distant 15 miles from a ship bears 40°. When will the point of land be abeam, if the ship moves north at 9 knots, and how far abeam?

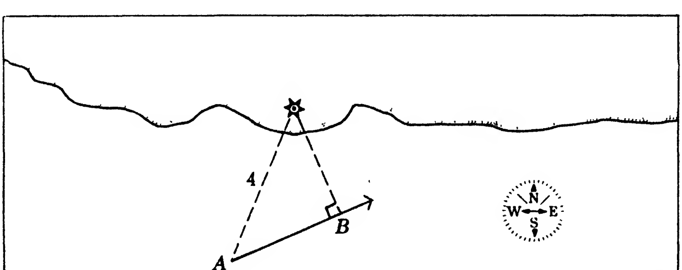


FIG. 2-41.

12. The navigator of a vessel steaming at 12 knots on course 65° observes a light bearing 22° true, distant 4 miles. Find the vessel's nearest approach to the light and the time interval between thè instant the vessel reached this position and the instant of observation. Figure 2-41 represents a chart on which the star symbol denotes the position of the light and AB denotes the course line of the vessel.

2-14. Vectors. A quantity which has both magnitude and direction is called a vector quantity. A **vector quantity** is represented by a directed straight-line segment called a **vector,** whose length is proportional to the magnitude and whose direction is indicated by an arrowhead at the end of the line.

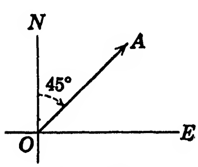


FIG. 2-42.

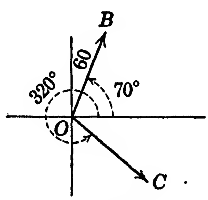


FIG. 2-43.

Thus in Fig. 2-42 vector OA completely describes the velocity of a vessel sailing northeast at a speed of 20 knots. OA is drawn to a convenient scale.

Likewise, in Fig. 2-43, the vector OB represents a force of 60 lb. pulling on a body at O at an angle of 70° from the horizontal; vector OC represents a force of 35 lb. acting on the same body at an angle of 320°.

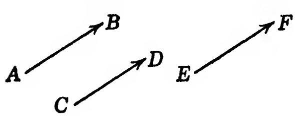


FIG. 2-44.

Two vectors are equal, if they have the same magnitude and the same direction. Thus in Fig. 2-44, vectors AB, CD, and EF are equal.

2-15. Addition of vectors. In Fig. 2-45 vector OA represents a motion from O to A and vector AB represents a motion from A to B.

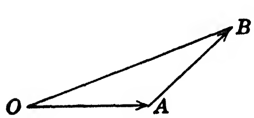


FIG. 2-45.

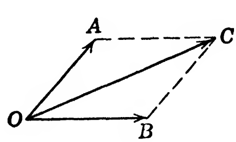


FIG. 2-46.

Then the sum of the two vectors represents the motion from O to B, or the vector OB. Thus, **the sum of two vectors is the vector joining the initial point of the first to the terminal point of the second, if the second vector begins at the end of the first.** If the two vectors start at the same point, as in Fig. 2-46, then their sum is represented by the diagonal of the

parallelogram of which the vectors are adjacent sides. Since the opposite sides of a parallelogram are parallel and equal, BC may be used as the vector in place of OA. This is known as the **parallelogram law** for the **composition of forces.** The diagonal of the parallelogram is called the **resultant** of the two forces. The resultant will produce the same effect on an object as the joint action of the two forces.

2-16. Components of a vector. A vector may be resolved along any two specified directions into two vectors of which it is the resultant. The two vectors are called the **components** of the first vector. If the components are at right angles to each other, they are called **rectangular components.** Thus in Fig. 2-47, a vector OC with a magnitude of 5 makes an angle of 53°6′ with the horizontal. By completing the rectangle $OBCA$, we get the **horizontal component** OA and the **vertical component** OB. Solving for OA and AC in triangle OAC, we find the magnitude of the components to be 3 and 4, respectively.

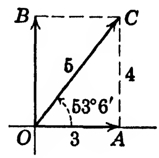


FIG. 2-47.

Example 1. Find the resultant of two forces of 50 lb. and 60 lb. acting at right angles to each other. What is the direction of the resultant with respect to the horizontal?

Solution. Figure 2-48 shows the relation between the forces and the resultant.

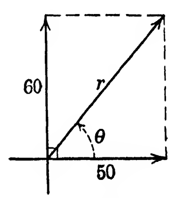


FIG. 2-48.

$$\tan\theta = \tfrac{60}{50} = 1.2000.$$
$$\therefore\ \theta = \mathbf{50°12'}.$$
$$\cos 50°12' = \frac{50}{r} \qquad \text{or} \qquad r = \frac{50}{\cos 50°12'}$$
$$\therefore\ r = \mathbf{78.1.}$$

Example 2. An airplane is flying on a course of 32° at a speed of 250 miles per hour. How many miles per hour is the plane advancing in a direction due east? in a direction due west? (The course is measured from the north.)

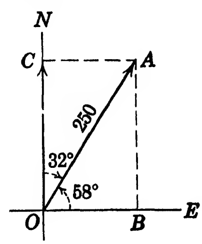


FIG. 2-49.

Solution. In triangle OBA (Fig. 2-49) $\dfrac{OB}{OA} = \cos 58°$, or

$$\begin{aligned} OB &= OA \cos 58° \\ &= 250(.5299) \\ &= \mathbf{132.47}, \text{ or } \mathbf{132\ m.p.h.} \end{aligned}$$

Also, $\dfrac{AB}{OA} = \sin 58°$, or

$$\begin{aligned} AB &= OA \sin 58° \\ &= 250(.8480) \\ &= \mathbf{212\ m.p.h.} \end{aligned}$$

EXERCISES 2-13

1. Find the horizontal and the vertical components of the following vectors with the given magnitudes and acting at the given angle with the horizontal:

(*a*) 25 at 35°. (*b*) 105 at 26°.
(*c*) 14.3 at 49°6′. (*d*) 20.6 at 56°30′.

2. An airplane flies a course of 54° at a speed of 220 knots. What is its eastward velocity? What is its northward velocity?

3. A boat heads directly north across a stream at a speed of 10 miles per hour. The water is flowing east at a speed of 4 miles per hour. What is the resultant speed of the boat? What is the direction of the path of the boat?

4. A cable supporting a captive balloon makes an angle of 70° with the ground. If the pull on the cable is 2500 lb., what is the vertical lifting force acting on the balloon and the horizontal force of the wind?

5. A boat is sailing east at the rate of 16 miles per hour. A man walks north across the deck at 3 miles per hour. Find his speed and the direction of his motion.

6. One force of 10 lb. is directed due east. The magnitude of a second force, directed due north is 14 lb. Find the magnitude and the direction of the resultant.

7. A pull of 400 lb. is applied to a cart at an angle of 18° to the horizontal. What is the effective horizontal pull? What is the effective vertical pull?

8. An automobile weighing 3500 lb. stands on a hill inclined 20°35′ from the horizontal. How large a force must be counteracted by the brakes of the automobile to prevent it from rolling downhill?

9. A 200-lb. shell for a battery is dragged up a runway inclined 35° to the horizontal. How much pressure does the shell exert against the runway? What force is required to drag the shell?

10. Find the force that is exactly sufficient to keep a 1200-lb. weight from sliding down a plane inclined 40° to the horizontal.

11. An airplane is heading north flying at 210 miles per hour. The wind is blowing from the west at 35 miles per hour. At the end of 45 min., how much distance has the airplane covered and in what direction is it flying?

12. If a gunboat starting from a point A steams 40 miles on course 135° to point B and then steams 30 miles on course 45°, find its bearing and distance from A.

13. To enter a harbor, a destroyer must pass from point A to point B, where vector AB runs 10 miles northwest. Due to the nature of the channel, the pilot runs north 4 miles, then northwest 5 miles, and then straight to B. Find the course of the last run and the total distance covered.

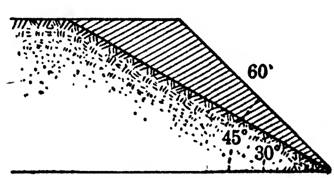
MISCELLANEOUS EXERCISES 2-14

Solve the following right triangles:

1. $a = 104$, $c = 185$.	**2.** $b = 47.78$, $B = 39°22'$.	**3.** $c = 5.890$, $B = 67°8'$.
4. $c = 625$, $A = 44°$.	**5.** $a = 4997$, $B = 62°44'$.	**6.** $a = 4.001$, $b = 7.923$.

7. Two straight roads cross at an angle of 52°36′, and there is a town on one road 6520 yd. from the crossing. What is the shortest distance from this town to the other road?

8. The Pennsylvania Railroad found it necessary, owing to land slides upon the roadbed, to reduce the angle of inclination of one bank of



Cross section

Fig. 2-50.

a certain railway cut near Pittsburgh, Pa., from an original angle of 45° to a new angle of 30°, as shown in Fig. 2-50. The bank as it originally

stood was 200 ft. long and had a slant length of 60 ft. Find the amount of the earth removed if the top level of the bank remained unchanged.

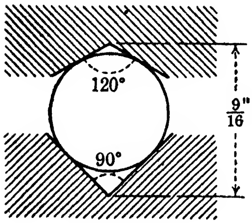


Fig. 2-51.

9. A slide in a machine is to run on rolling balls. The balls run in grooves with straight sides as shown in Fig. 2-51. The angle of the upper (moving) groove is 120°, and that of the lower (fixed) groove is 90°. What size of balls should be used?

10. A searchlight situated on a straight coast has a range of 43 miles. A ship sails on a line parallel to the coast and 15 miles from it. What is the distance covered by the ship while it remains within range of the light? What angle is subtended at the light by a line connecting the extreme positions of the ship?

11. A man in a balloon observes that the straight line connecting the bases of two towers, which are 1 mile apart on a horizontal plane, subtends an angle of 70°. If he is exactly above the middle point of this line, find the height of the balloon.

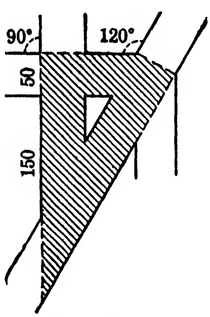


Fig. 2-52.

12. Find the number of square feet of pavement required for the shaded portion of the streets shown in Fig. 2-52, all the streets being 50 ft. wide.

13. A flagstaff 25 ft. high stands on the top of a house. From a point on the plane on which the house stands, the angles of elevation of the top and the bottom of the flagstaff are observed to be 60° and 45°, respectively. Find the height of the house.

14. From a point A, 10 ft. above the water, the angle of elevation of the top of a lighthouse is 46°, and the angle of depression of its image in the water is 50°. Find the height h of the lighthouse and its horizontal distance from the observer (see Fig. 2-53).

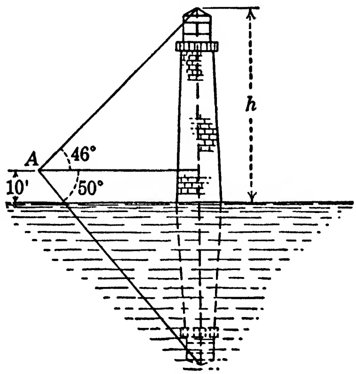


FIG. 2-53.

15. The pilot in an airplane observes the angle of depression of a light directly below his line of flight to be 30°. A minute later its angle of depression is 45°. If he is flying horizontally in a straight course at the rate of 150 miles per hour, find (*a*) the altitude at which he is flying; (*b*) his distance from the light at the first point of observation.

16. From the top of a building the angle of depression of a point in the same horizontal plane with the base of the building is observed to be 47°13′. What will be the angle of depression of the same point when viewed from a position halfway up the building?

17. The captive balloon shown in Fig. 2-54 is connected to a ground station A by a cable of length 842 ft. inclined 65° to the horizontal. In a vertical plane with the balloon and its station and on the opposite side of the balloon from A, a target B is sighted from the balloon on a level with A. If the angle of elevation of the balloon from the target is 4°, find the distance from the target to a point C directly under the balloon.

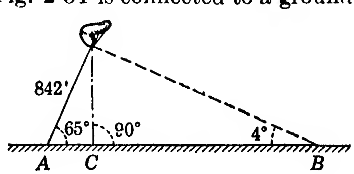


FIG. 2-54.

18. A straight line AB on the side of a hill is inclined at 15° to the horizontal. The axis of a tunnel 486 ft. long is inclined 28°25′ below the horizontal and lies in a vertical plane with AB. How long is a vertical hole from the bottom of the tunnel to the surface of the hill?

19. A lighthouse standing on the top of the cliff shown in Fig. 2-55 is observed from two boats A and B in a vertical plane through the lighthouse. The angle of elevation of the top of the lighthouse viewed from B is 16°, and the angles of elevation of the top and bottom viewed from A are 40° and 23°, respectively. If the boats are 1320 ft. apart, find the height of the lighthouse and the height of the cliff.

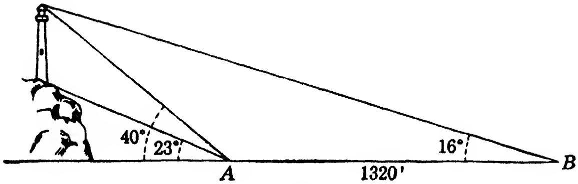


FIG. 2-55.

20. The church A and the lighthouse B represented in Fig. 2-56 were observed from a ship at point S to be on a straight line passing through S

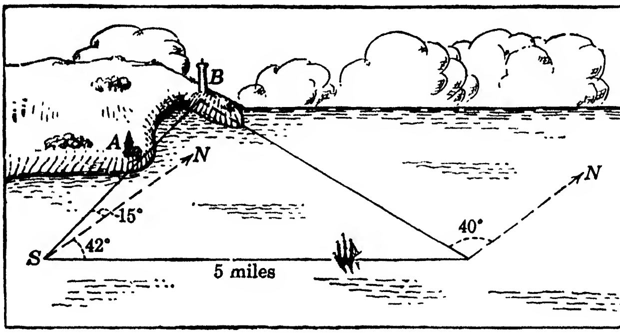


FIG. 2-56.

and bearing N. 15° W. After sailing 5 miles on a course N. 42° E., the captain of the ship found that A bore due west and B bore N. 40° W. Find the distance from the church to the lighthouse.

21. A tower (Fig. 2-57) of height h stands on level ground and is due north of point A and due east of point B. At A and B the angles of

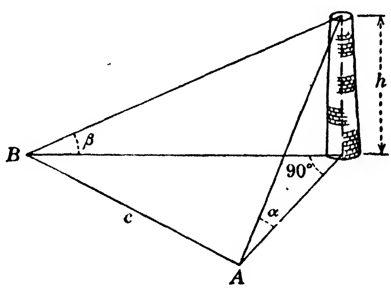


FIG. 2-57.

elevation of the top of the tower are α and β, respectively. If the distance AB is c, show that

$$h = \frac{c}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}.$$

22. Given the oblique triangle ABC of Fig. 2-58 in which A, B, and a are known. Show that $b = \frac{a}{\sin A} \sin B$.

Hint. Drop a perpendicular p from the vertex C to the side AB. Find two values of p and equate them.

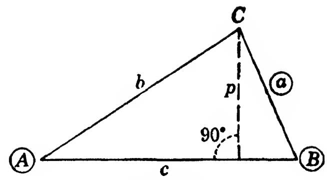


Fig. 2-58.

23. In the oblique triangle ABC (shown in Fig. 2-59) prove that $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

Hint. $AD = b \cos A$, and $DB = c - b \cos A$. Equate two values of p.

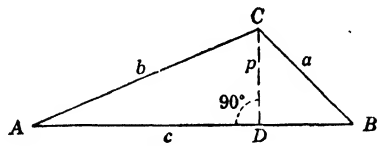


Fig. 2-59.

24. If R is the radius of a circle, show that the area of a regular circumscribed polygon of n sides is $A = nR^2 \tan \frac{180°}{n}$.

25. Show that the area of a regular polygon of n sides each of length a is given by $A = \frac{na^2}{4} \cot \frac{180°}{n}$.

26. A ship asks bearings from two radio stations A and B. A reports the ship's bearing 82° and B reports 127°. Station B is known to be 127 nautical miles from A on bearing 58° from A. Find the difference in latitude and the departure of the ship from A.

27. The relative positions of point A at the bow of a steamer 326 ft. long, C at its stern, and B on a near-by submarine are shown in Fig. 2-60.

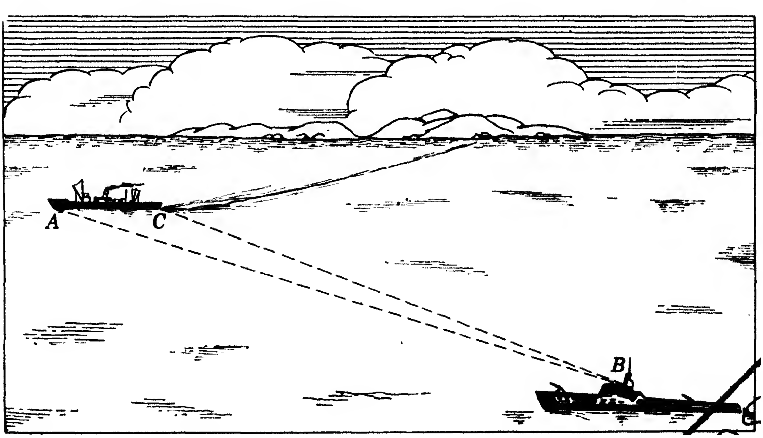


Fig. 2-60.

If angles ABC and ACB are 18° and 90°, respectively, about how far is the submarine from the steamer?

28. A ship is 320 ft. long and subtends an angle of 10° at a boat equidistant from the ends of the ship. How far is the boat from the ship?

29. Two ships start from the same point. One sails southwest 27.5 miles and the other northwest 33.5 miles. How far apart are they?

30. A submerged rock is 3650 ft. from a lighthouse 208 ft. high. A sailor in a boat wishes to pass the rock and be 3040 ft. from it when it is between him and the lighthouse. What angle of elevation for the top of the lighthouse should his sextant register?

31. A line connecting two buoys A and B 6080 ft. apart runs northeast from A to B. From a submarine, A bears 305° and B bears 35°. How far is the submarine from B?

32. In what direction should a ship head in order to reach a point 8 miles due north of her, if her speed in still water is 10 knots and a 2-knot current is running due east?

33. The pilot on a tanker moving north observes a lighthouse bearing 54°. After running 40 min. at 9 knots, it is abeam and the angle of elevation of its top is 24′. If its base is at the same height as the observer, how high is the lighthouse?

34. The water within 1500 yd. of a lighthouse rising 200 ft. above sea level is too shallow for navigation, but there is no danger at a distance of 1800 yd. Find the "danger angle" corresponding to the 1800-yd.

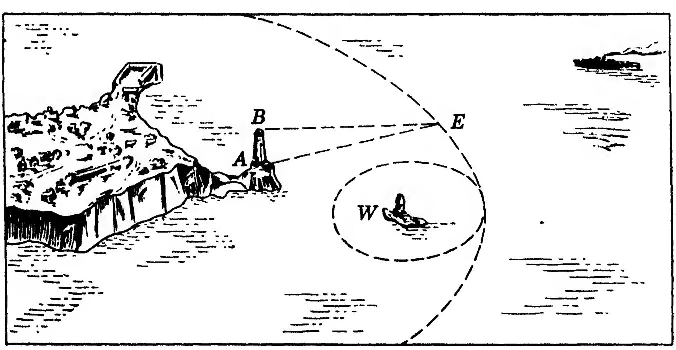


FIG. 2-61.

distance (see Fig. 2-61). The danger angle is the angle of elevation of the lighthouse from a point on the water on the edge of the danger circle.

35. A destroyer is 10 miles due south of a tanker. The destroyer steams at 11 knots on course 27° while the tanker on course 117° at 10 knots. Find the distance one will pass ahead of the other.

36. A ship steaming north observes two buoys bearing east. After a 3-mile run one buoy bears 135° and the farther one 117°. Find the distance between them.

37. An observer sees the top of a lighthouse 82 ft. high in line with the top of another 60 ft. farther away. If his distance from the nearer one is 120 ft., find the height of the other.

38. A beacon 122 ft. high was north of a submarine and 4 miles away from it. Find the angle of elevation of the beacon from the submarine after it had sailed 4 miles due east.

39. The angle of elevation of beacon A from a tug is 54′ and that of beacon B is 1°12′. B, which is at the same height as A, bears 10° as viewed from the tug and 100° as viewed from A. Find the course of the tug to reach A.

40. Two towers of equal height and 3.78 miles apart subtend a horizontal angle of 40°35′ at a ship and have equal angles of elevation from her. Find her distance from a point on the water midway between the towers.

CHAPTER 3

FUNDAMENTAL RELATIONS AMONG THE TRIGONOMETRIC FUNCTIONS

3-1. Introduction. In this chapter the student should become familiar with some important elementary relations connecting the trigonometric functions and learn to use them with facility. Since one value of a trigonometric function of an acute angle determines the angle and since there are six such functions, we shall find many relations among them. Among the forms of expressing them there is usually one best adapted to our purposes. To obtain this one, it is often convenient to use a number of elementary identities.

3-2. Reciprocal relations. For convenience, we shall recall the reciprocal relations discussed in Art. 1-5.

$$\left.\begin{aligned} \sin A &= \frac{1}{\csc A}, & \csc A &= \frac{1}{\sin A}, \\ \cos A &= \frac{1}{\sec A}, & \sec A &= \frac{1}{\cos A}, \\ \tan A &= \frac{1}{\cot A}, & \cot A &= \frac{1}{\tan A}. \end{aligned}\right\} \tag{1}$$

3-3. Fractional relations. In triangle ABC in Fig. 3-1,

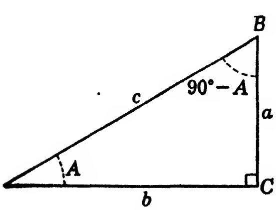


FIG. 3-1.

$$\tan A = \frac{a}{b} = \frac{a/c}{b/c} = \frac{\sin A}{\cos A},$$

$$\cot A = \frac{b}{a} = \frac{b/c}{a/c} = \frac{\cos A}{\sin A}.$$

Therefore

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}. \tag{2}$$

3-4. Complementary relations. Another set of equations has reference to complementary angles. In triangle ABC in Fig. 3-1,

$$\sin A = \frac{a}{c} \qquad \text{and} \qquad \cos (90° - A) = \frac{a}{c}.$$

Since $\sin A$ and $\cos (90° - A)$ are both equal to a/c, we have

$$\sin A = \cos (90° - A).$$

By using the same kind of argument in connection with each of the trigonometric functions, the student may prove the following equations:

$$\left.\begin{aligned} \cos (90° - A) &= \sin A, & \sin (90° - A) &= \cos A, \\ \cot (90° - A) &= \tan A, & \tan (90° - A) &= \cot A, \\ \csc (90° - A) &= \sec A, & \sec (90° - A) &= \csc A, \end{aligned}\right\} \qquad (3)$$

or, stated in other words, **any trigonometric function of an acute angle is equal to the co-function of its complement.** This statement shows the significance of the prefix **co-** in the names of the trigonometric functions; it has reference to the word *complement.*

3-5. The relations derived from a figure. From Fig. 3-2 we read

$$\frac{a}{1} = \sin A, \qquad \text{or} \qquad a = \sin A,$$

$$\frac{b}{1} = \cos A, \qquad \text{or} \qquad b = \cos A.$$

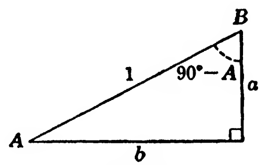


FIG. 3-2.

By replacing a by $\sin A$ and b by $\cos A$ in Fig. 3-2, we obtain Fig. 3-3. Now apply the definitions of the trigonometric functions to read, from Fig. 3-3,

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A}, \quad (4)$$

$$\sec A = \frac{1}{\cos A}, \qquad \csc A = \frac{1}{\sin A}. \quad (5)$$

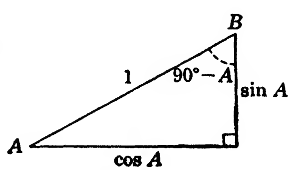


FIG. 3-3.

Using (4) we obtain

$$\cot A = \frac{\cos A}{\sin A} = 1 \div \frac{\sin A}{\cos A} = \frac{1}{\tan A}. \qquad (6)$$

Next read the functions of $(90° - A)$ from Fig. 3-3 to get $\sin (90° - A) = \cos A$, $\cos (90° - A) = \sin A$, and the other relations of (3). Since one may obtain the relations (1), (2), and

(3) directly from Fig. 3-3, it is only necessary to draw the figure to recall them.

3-6. Identities and conditional equations. An identity is an equation that is true for all values of the variables for which its members are defined. Thus the equations

$$1 - x^2{}^* \equiv (1 - x)(1 + x), \qquad \csc x \equiv \frac{1}{\sin x},$$

are true for all values of x for which they are defined and are therefore identities. The equation $x^2 = 1$ is not an identity, since it is true only when $x = 1$ or -1. Similarly $\sin x = \cos x$ is a conditional equation, since 45° is the only acute angle for which it is true. Equations (1), (2), and (3) of this chapter are identities. Familiarity with these identities will be obtained by using them to simplify expressions, to verify identities, to find solutions of equations of condition, and to solve various kinds of problems.

Example 1. Simplify

$$\sin A \cos (90° - A) \csc A \cot A - \sin (90° - A). \qquad (a)$$

Solution. From equations (3), we have

$$\cos (90° - A) = \sin A, \qquad \sin (90° - A) = \cos A, \qquad (b)$$

and from equations (1) and (2)

$$\csc A = \frac{1}{\sin A}, \qquad \cot A = \frac{\cos A}{\sin A}. \qquad (c)$$

Replacing $\cos (90° - A)$, $\sin (90° - A)$, $\cot A$, and $\csc A$ in (a) by their values from (b) and (c), we obtain

$$\sin A \cdot \sin A \cdot \frac{1}{\sin A} \cdot \frac{\cos A}{\sin A} - \cos A. \qquad (d)$$

Since $\sin A$ is a number it may be canceled with $\sin A$. Hence (d) simplifies to

$$\cos A - \cos A = \mathbf{0}.$$

* The symbol $\equiv$ is frequently used to mean "is identically equal to." However, for convenience, we shall use the ordinary symbol of equality throughout the book.

Example 2. Find an acute angle that satisfies the equation

$$\tan B = \cot 2B.$$

Solution. Using one of the equations of (3) to replace $\cot 2B$ by $\tan (90° - 2B)$, we obtain $\tan B = \tan (90° - 2B)$. Hence, $B = 90° - 2B$. Solving this equation, we find $\boldsymbol{B} = \mathbf{30°}$.

Example 3. Find an acute angle x that satisfies the equation

$$\sin (3x - 30°) = \cos (2x + 10°).$$

Solution. Using the first equation of (3) to replace

$$\cos (2x + 10°)$$

by $\sin (90° - 2x - 10°)$, we obtain

$$\sin (3x - 30°) = \sin (90° - 2x - 10°).$$

This equation is satisfied if

$$3x - 30° = 90° - 2x - 10°.$$

Solving this equation for x, we get $\boldsymbol{x} = \mathbf{22°}$.

EXERCISES 3-1

1. Express as trigonometric functions of angles less than 45°

(*a*) $\sin 75°$. (*b*) $\cos 87°$. (*c*) $\tan 89°30'$.
(*d*) $\sec 49°20'$. (*e*) $\cot 45°50'$. (*f*) $\csc 70°40'$.

2. Find for each of the following equations an acute angle that satisfies it:

$$\sin (2x - 20°) = \cos (3x + 10°).$$
$$\cos (5\theta - 10°) = \sin (3\theta + 20°).$$
$$\tan (65° - 3\theta) = \cot (5° + 7\theta).$$
$$\csc (2\theta + 70°) = \sec (4\theta - 36°).$$

3. Simplify

(*a*) $\sin \theta \cot \theta$. (*b*) $\cos \theta \tan \theta$. (*c*) $\sec \theta \cot \theta$.

(*d*) $\cos (90° - \theta) \sec \theta \cot \theta$.
(*e*) $\csc \theta \cot (90° - \theta)$.
(*f*) $\sin \theta \cos (90° - \theta) \csc \theta \tan (90° - \theta)$.
(*g*) $(\tan \theta)^2 (\cos \theta)^2 (\csc \theta)^2$.
(*h*) $(\cot \theta)^2 [\cos (90° - \theta)]^2 (\sec \theta)^2$.
(*i*) $\sin \theta \cos (90° - \theta) \tan (90° - \theta) (\sec \theta)^2$.

4. Draw Fig. 3-3, and apply the definitions of the trigonometric functions to read from it all six functions of A and of $90° - A$. Compare the result with equations (1), (2), and (3).

5. Verify each of the following identities by transforming the left-hand member, the right-hand member, or both members until they have the same form:

(*a*) $1 + \sin\alpha \cot\alpha = \sin\alpha \csc\alpha + \cos\alpha.$
(*b*) $\tan\alpha + \sec\alpha = \sin\alpha \csc(90° - \alpha) + \tan\alpha \csc\alpha.$
(*c*) $(\sin\alpha)^2 \csc\alpha \cot\alpha - \cos\alpha = (\cos\alpha)^2 \sec\alpha \tan\alpha - \sin\alpha.$
(*d*) $\dfrac{(\sin\theta)^2}{(\cos\theta)^2} = (\sin\theta)^4 (\sec\theta)^2 (\csc\theta)^2.$
(*e*) $\dfrac{\cot\theta}{\csc\theta} = \sin(90° - \theta).$
(*f*) $\cos\varphi \csc\varphi \tan\varphi = 1.$
(*g*) $(\sin A)^2 (\csc A)^2 + (\cos A)^2 (\sec A)^2 = 2.$
(*h*) $\dfrac{\cos A \tan A}{\tan(90° - A)} = (\sin A)^2 \sec A.$
(*i*) $\tan\theta (\cos\theta)^2 - \tan(90° - \theta)(\sin\theta)^2 = 0.$
(*j*) $\sin\theta \tan\theta \sec\theta = \sec\theta \cot(90° - \theta) \sin\theta.$
(*k*) $\sec\theta \cot\theta \cot(90° - \theta) - \sin\theta \csc(90° - \theta) = \sec\theta - \tan\theta.$
(*l*) $\tan(3\theta) = \dfrac{\sec(3\theta)}{\csc(3\theta)}.$
(*m*) $\tan(3\theta)\tan(90° - 3\theta) + \sin(2\theta)\csc(2\theta) + \cos\theta \sec\theta = 3.$

6. For each of the following equations find an acute angle that satisfies it:

$$\tan(6\theta - 50°)\tan(57° + \theta) = 1.$$
$$\sin(9\theta + 10°12')\sec(2\theta + 8°40') = 1.$$
$$\csc(4\theta + 43°29')\cos(5\theta + 5°13') = 1.$$
$$\tan(8\theta - 35°)\sin(2\theta - 22°) = \cos(2\theta - 22°).$$

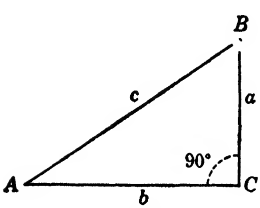


FIG. 3-4.

3-7. Relations derived from the Pythagorean theorem. From the right triangle ABC of Fig. 3-4 we have, by the well-known Pythagorean theorem,

$$a^2 + b^2 = c^2. \tag{7}$$

Dividing both members of this equation first by c^2, then by b^2, and finally by a^2, we obtain

$$\left.\begin{aligned}\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 &= \left(\frac{c}{c}\right)^2,\\ \left(\frac{a}{b}\right)^2 + \left(\frac{b}{b}\right)^2 &= \left(\frac{c}{b}\right)^2,\\ \left(\frac{a}{a}\right)^2 + \left(\frac{b}{a}\right)^2 &= \left(\frac{c}{a}\right)^2.\end{aligned}\right\} \tag{8}$$

Expressing the quantities inside the parentheses in terms of trigonometric functions of the angle A, we have

$$\left.\begin{aligned}\sin^2 A + \cos^2 A &= 1,\\ \tan^2 A + 1 &= \sec^2 A,\\ 1 + \cot^2 A &= \csc^2 A,\end{aligned}\right\} \tag{9}$$

where $\sin^2 A$ means $(\sin A)^2$, $\cos^2 A$ means $(\cos A)^2$, etc.

Another method of deriving these formulas consists of applying the Pythagorean theorem to Fig. 3-5 to obtain

$$\sin^2 A + \cos^2 A = 1$$

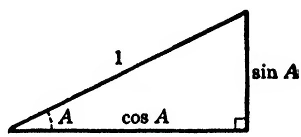


FIG. 3-5.

and then dividing this equation first by $\cos^2 A$ and then by $\sin^2 A$ to obtain

$$\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A},$$

or

$$\tan^2 A + 1 = \sec^2 A,$$

and

$$\frac{\sin^2 A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A},$$

or

$$1 + \cot^2 A = \csc^2 A.$$

EXERCISES 3-2

1. By using relations (9) simplify

(a) $1 - \sin^2 \beta$. (b) $1 - \cos^2 \beta$. (c) $\sec^2 \beta - 1$.
(d) $\sec^2 \beta - \tan^2 \beta$. (e) $1 - \csc^2 \beta$. (f) $\csc^2 \beta - \cot^2 \beta$.
(g) $\dfrac{\sin^2 A + \cos^2 A}{\sec^2 A - \tan^2 A}$. (h) $\dfrac{1 - \cos^2 \theta}{1 - \csc^2 \theta}$.

2. Use equations (1), (2), (3), and (9) to simplify

(a) $\dfrac{\sin^2 \varphi + \cos^2 \varphi}{\sec \varphi \cos \varphi}$. (b) $(\sec^2 \varphi - 1)(\csc^2 \varphi - 1)$.

(c) $\dfrac{(1 - \sin \varphi)(1 + \sin \varphi)}{(1 - \cos \varphi)(1 + \cos \varphi)}$. (d) $\tan \varphi + \cot \varphi$.

(e) $\dfrac{\sin \varphi}{\csc \varphi} + \dfrac{\cos \varphi}{\sec \varphi}$. (f) $(\sin \varphi + \cos \varphi)^2 - 2 \sin \varphi \cos \varphi$.

3. Transform each of the following expressions so that the equivalent expression will contain only sines and cosines of θ, then replace $\cos \theta$ by $\sqrt{1 - \sin^2 \theta}$ so that the final expression will contain no trigonometric functions except $\sin \theta$:

(a) $2 \sin \theta \cos^4 \theta \tan^2 \theta$. (b) $\dfrac{\tan^2 \theta - 1}{\tan^2 \theta + 1}$.

(c) $\cos^4 \theta - \sin^4 \theta$. (d) $(\tan \theta - \cot \theta) \sin \theta \cos \theta$.

(e) $\sec \theta - \sin^2 \theta \sec^2 \theta$. (f) $\tan \theta \sec^2 \theta - \cot (90° - \theta)$.

4. Transform each of the expressions in the left-hand column into the one written to the right of it.

(a) $\csc^2 \theta + \sec^2 \theta$.	$\sec^2 \theta \csc^2 \theta$.
(b) $\dfrac{1}{\tan^2 A + 1} + \dfrac{1}{\cot^2 A + 1}$.	1.
(c) $\cos \theta \tan \theta$.	$\sin \theta$.
(d) $\sin^2 \theta \div \csc^2 \theta$.	$\sin^4 \theta$.
(e) $\dfrac{\cot^2 A}{1 + \cot^2 A}$.	$\cos^2 A$.
(f) $\cos^2 A \tan^2 A + \sin^2 A \cot^2 A$.	1.
(g) $1 + \dfrac{\tan^2 A}{1 + \sec A}$.	$\sec A$.

3-8. Verification of identities. There are two methods of procedure for verifying identities. By means of the fundamental identities* and suitable algebraic operations, (a) *the more complicated member of the identity may be transformed into the other*

* Although we have proved the identities (1), (2), (3), and (9) only for acute angles, they will be found to be true, as soon as we have defined the trigonometric functions of the general angle, for all angles for which the functions are defined. A similar statement applies to all the identities of this article.

member of the identity; (b) both members may be transformed into the same expression. It may be advisable, as a last resort, to transform both members into expressions that contain only one trigonometric function. The following examples will illustrate methods of procedure:

Example 1. Verify the identity

$$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta.$$

Verification. Expansion of the left-hand member gives

$$\tan^2\theta + 2\tan\theta\cot\theta + \cot^2\theta.$$

Since $\cot\theta \cdot \tan\theta = 1$, we may write this in the form

$$\tan^2\theta + 2 + \cot^2\theta,$$

or

$$(\tan^2\theta + 1) + (1 + \cot^2\theta).$$

From the last two equations of (9), this expression is

$$\sec^2\theta + \csc^2\theta.$$

Example 2. Verify the identity

$$1 - \cot^4\theta = 2\csc^2\theta - \csc^4\theta.$$

Verification. In the following outline, the work on the left of the vertical line gives the steps for reducing the left-hand member to a function of $\sin\theta$; the work on the right of the vertical line applies to the right-hand member:

$$\begin{array}{c|c}
1 - \cot^4\theta = & 2\csc^2\theta - \csc^4\theta = \\
1 - \dfrac{\cos^4\theta}{\sin^4\theta} = & \dfrac{2}{\sin^2\theta} - \dfrac{1}{\sin^4\theta} = \\
\dfrac{\sin^4\theta - \cos^4\theta^*}{\sin^4\theta} = & \dfrac{2\sin^2\theta - 1}{\sin^4\theta}. \\
\dfrac{\sin^4\theta - (1 - \sin^2\theta)^2}{\sin^4\theta} = & \\
\dfrac{-1 + 2\sin^2\theta}{\sin^4\theta}. &
\end{array}$$

* Beginning at this point we could have written

$$(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta) = \sin^2\theta - (1 - \sin^2\theta) = 2\sin^2\theta - 1.$$

Thus the identity is verified, since we have shown that both its members are equal to the same expression.

Alternative verification. The steps outlined in the following plan give a more direct verification:

$1 - \cot^4 \theta =$	$2 \csc^2 \theta - \csc^4 \theta =$
$(1 + \cot^2 \theta)(1 - \cot^2 \theta) =$	$\csc^2 \theta(2 - \csc^2 \theta) =$
$\csc^2 \theta(1 - \cot^2 \theta).$	$\csc^2 \theta(2 - \cot^2 \theta - 1) =$
	$\csc^2 \theta(1 - \cot^2 \theta).$

EXERCISES 3-3

Simplify each of the following expressions:

1. $\tan x \sin x + \cos x.$
2. $\cot A - \sec A \csc A(1 - 2 \sin^2 A).$
3. $(\tan B + \cot B) \sin B \cos B.$
4. $\tan A \sin A \cos A + \sin A \cos A \cot A.$
5. $(\cot^2 A - \csc^2 A)(\sec^2 A - \tan^2 A).$
6. $(\cos^2 \theta - 1) \csc^2 \theta.$

Transform each of the following expressions into the expression written to the right of it:

7. $\cos \theta \csc \theta \tan \theta.$	$1.$
8. $\tan A \sec A \cot A \cos A \tan (90° - A).$	$\cot A.$
9. $\csc A \cot A \cos A + 1.$	$\csc^2 A.$
10. $\dfrac{1}{\sin^2 A} + \dfrac{1}{\cos^2 A}.$	$\sec^2 A \csc^2 A.$
11. $\sec^2 A \csc^2 A.$	$\sec^2 A + \csc^2 A.$
12. $(\sec \theta - \cos \theta)(\csc \theta - \sin \theta).$	$\sin \theta \cos \theta.$
13. $(\sec A - \tan A)(\sec A + \tan A).$	$1.$
14. $(\csc A - \cot A)(\csc A + \cot A).$	$1.$
15. $\sin (90° - B) \cot B \sin B - 1.$	$- \sin^2 B.$
16. $2 \cos^2 A - 1.$	$1 - 2 \sin^2 A.$
17. $\sec^2 A + \tan^2 A.$	$2 \sec^2 A - 1.$

Verify the following identities:

18. $\sin \theta \sec \theta \cot \theta = 1.$
19. $(\tan y + \cot y) \cot y = \csc^2 y.$
20. $\tan A = \dfrac{\sec A}{\csc A}.$
21. $(\cos A - 1)(\cos A + 1) = - \sin^2 A.$
22. $\cot C \sin C + \cos C = 2 \cos C.$

23. $\tan(90° - A) \tan A - \cos^2(90° - A) = \sin^2(90° - A)$.

24. $\sin\theta \cot\theta + \cos^2\theta \sec\theta = 2\cos\theta$.

25. $\cos^2\alpha(1 + \tan^2\alpha) = 1$.

26. $\cot\theta \cos\theta + \sin\theta = \csc\theta$.

27. $\sin^2 A \sec^2 A = \sec^2 A - 1$.

28. $(\sin\varphi - \cos\varphi)^2 = 1 - 2\sin\varphi\cos\varphi$.

29. $\dfrac{\cos\beta}{1 + \sin\beta} + \dfrac{\cos\beta}{1 - \sin\beta} = 2\sec\beta$.

30. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$.

31. $(1 - \sec^2 A)(1 - \csc^2 A) = 1$.

32. $\dfrac{1 + \tan^2\alpha}{1 + \cot^2\alpha} = \tan^2\alpha$.

33. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x} = 2\sec x$.

34. $\csc^2\varphi - \csc^2\varphi\cos^2\varphi = 1$.

35. $\tan x + \cot x = \sec x \csc x$.

36. $(\cot\alpha - \tan\alpha)^2 \sin^2\alpha\cos^2\alpha = 1 - 4\sin^2\alpha\cos^2\alpha$.

37. $\sec^4\alpha - \tan^4\alpha = \sec^2\alpha + \tan^2\alpha$.

38. $\dfrac{\sec A + \csc A}{\sin A + \cos A} = \sec A \csc A$.

39. $\dfrac{\csc\theta + 1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta - 1}$.

40. $\tan A \sin A + \cos A = \sec A$.

41. $\csc^4 A - \cot^4 A = 2\cot^4 A + 1$.

42. $\dfrac{\tan x - \cot x}{\sin x - \cos x} = \sec x + \csc x$.

43. $\dfrac{\tan\theta\sin\theta}{\tan\theta - \sin\theta} = \dfrac{\sin\theta}{1 - \cos\theta}$.

44. $\dfrac{\cot B - \cos B}{\cos^3 B} = \dfrac{1 - \sin B}{\cos^2 B \sin B}$.

45. $\tan\varphi - \csc\varphi\sec\varphi(1 - 2\cos^2\varphi) = \cot\varphi$.

46. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A\cos^2 A$.

47. $\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \csc x - \cot x$.

48. $\sqrt{\dfrac{\sec\varphi - \tan\varphi}{\sec\varphi + \tan\varphi}} = \sec\varphi - \tan\varphi$.

49. $\dfrac{\sec y + \tan y}{\cos y + \cot y} = \sec y \tan y$.

50. $(\sec\theta + \tan\theta)^2 = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$

51. $\cot y + \dfrac{\sin y}{1 + \cos y} = \csc y.$

52. $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 - \sin A}{\cos A} = 2(\sec A - \tan A).$

53. $\dfrac{1}{(\cos^2 x - \sin^2 x)^2} - \dfrac{4\tan^2 x}{(1 - \tan^2 x)^2} = 1.$

54. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}.$

3-9. Formulas from right triangles. It appeared in Art. 3-5 that we could read formulas (1), (2), and (3) directly from Fig. 3-3. Other identities may be obtained in the same manner.

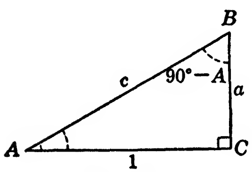


FIG. 3-6.

For example, we draw the right triangle shown in Fig. 3-6 with leg AC equal to 1. Then

$$\frac{a}{1} = \tan A,$$

$$\frac{c}{1} = \sec A.$$

Figure 3-7 is obtained by replacing a by $\tan A$ and c by $\sec A$ in Fig. 3-6. Using the definitions of the trigonometric functions on Fig. 3-7, we get

$$\cot A = \frac{AC}{CB} = \frac{1}{\tan A}, \qquad \cos A = \frac{AC}{AB} = \frac{1}{\sec A},$$

$$\cot(90° - A) = \frac{BC}{AC} = \tan A, \qquad \csc(90° - A) = \frac{AB}{AC} = \sec A.$$

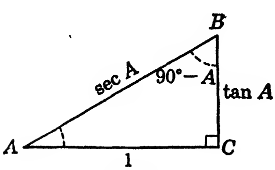


FIG. 3-7.

By applying the Pythagorean theorem to Fig. 3-7, we get

$$\mathbf{1 + \tan^2 A = \sec^2 A.} \qquad \textbf{(10)}$$

Evidently other identities could also be obtained. Thus, from Fig. 3-7, we read

$$\sin A = \frac{\tan A}{\sec A}, \qquad \cos(90° - A) = \frac{\tan A}{\sec A}, \text{ etc.}$$

Figure 3-8 was obtained by using the idea underlying the con-

struction of Fig. 3-7. From it we read

$$\tan A = \frac{1}{\cot A}, \qquad \sin A = \frac{1}{\csc A},$$
$$\tan B = \tan (90° - A) = \cot A,$$
$$\sec (90° - A) = \csc A,$$
$$\mathbf{1 + \cot^2 A = \csc^2 A,} \qquad \textbf{(11)}$$

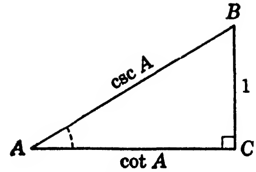


FIG. 3-8.

and others. The fundamental identities can be recalled at any time by reproducing Figs. 3-3, 3-7, and 3-8 and reading the identities directly from these figures.

By means of figures, it is a simple matter to express all the trigonometric functions in terms of one. Figure 3-9 is about the same as Fig. 3-7; instead of replacing AB by *sec* A, we have observed that

$$AB = \sqrt{\overline{AC}^2 + \overline{CB}^2} = \sqrt{1 + \tan^2 A}$$

and have written $\sqrt{1 + \tan^2 A}$ on AB. The definitions of the trigonometric functions may now be used to read from Fig. 3-9.

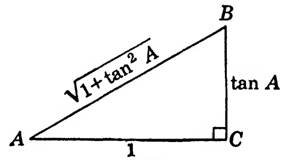


FIG. 3-9.

$$\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \qquad \cos A = \frac{1}{\sqrt{1 + \tan^2 A}},$$
$$\sec A = \sqrt{1 + \tan^2 A}, \qquad \csc A = \frac{\sqrt{1 + \tan^2 A}}{\tan A},$$
$$\cot A = \frac{1}{\tan A}.$$

EXERCISES 3-4

1. Using Fig. 3-10, express all the trigonometric functions of angle A in terms of sin A.

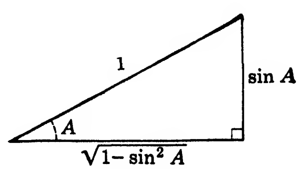


FIG. 3-10.

2. Using Fig. 3-11, express all the trigonometric functions of angle A in terms of cos A.

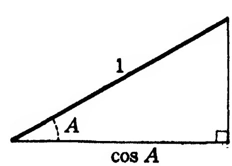


FIG. 3-11.

3. Express all the trigonometric functions of angle A in terms of (*a*) $\cot A$, (*b*) $\sec A$, (*c*) $\csc A$.

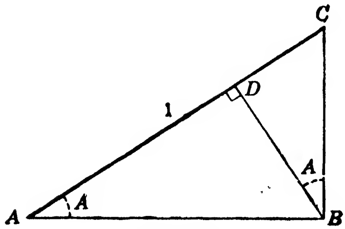


FIG. 3-12.

4. In Fig. 3-12 $AC = 1$. Find the lengths CB, AB, AD, and DC and equate two values of AC to obtain

$$\sin^2 A + \cos^2 A = 1.$$

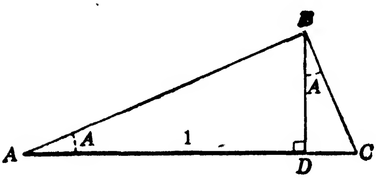


FIG. 3-13.

5. In Fig. 3-13 $AD = 1$. Find the lengths of AB, BD, AC, and CD and equate two values of AC to obtain

$$1 + \tan^2 A = \sec^2 A.$$

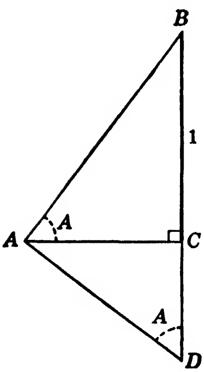


FIG. 3-14.

6. In Fig. 3-14 $BC = 1$. Find AB, BD, AC, and CD and equate two values of BD to obtain

$$1 + \cot^2 A = \csc^2 A.$$

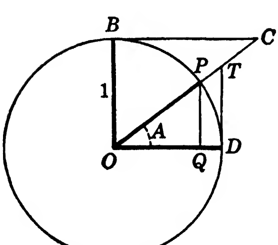


FIG. 3-15.

7. The radius of the circle in Fig. 3-15 is 1. Find the lengths of the line segments PQ, OQ, TD, OT, OC, BC, write them on the figure, and read from the figure the following identities:

$$\sin^2 A + \cos^2 A = 1,$$
$$1 + \tan^2 A = \sec^2 A,$$
$$1 + \cot^2 A = \csc^2 A.$$

3-10. Length of line segments. Consider the right triangle ABC in Fig. 3-16. The given parts A and c are encircled. First

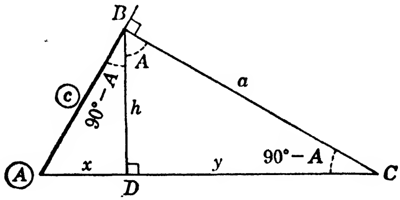


FIG. 3-16.

let us try to express x, h, y, and a in terms of the given parts. From triangle ABD, we write

$$\frac{x}{c} = \cos A; \qquad \therefore\ x = c \cos A. \tag{12}$$

$$\frac{h}{c} = \sin A; \qquad \therefore\ h = c \sin A. \tag{13}$$

Similarly, from triangle BDC, we have

$$\frac{y}{h} = \tan A; \qquad \therefore\ y = h \tan A. \tag{14}$$

Replacing h in this formula by its value $c \sin A$ from (13), we have

$$y = c \sin A \tan A. \tag{15}$$

Also from triangle BDC, we get

$$\frac{a}{h} = \sec A; \qquad \therefore\ a = h \sec A. \tag{16}$$

Replacing h in this formula by its value $c \sin A$ from (13), we have

$$a = c \sin A \sec A. \tag{17}$$

Figure 3-17 is obtained from Fig. 3-16 by replacing x, y, h, and a by their values from (12), (14), (13), and (17), respectively.

It is to be observed that when there are given only enough parts of a rectilinear figure to determine it and when all parts of the figure have been expressed in terms of the given ones, then any relation obtained by reading an equation from the figure, either by applying a proposition from geometry or by

using the definitions of the trigonometric functions, is an identity. Thus an identity may be formed from Fig. 3-17 by using the

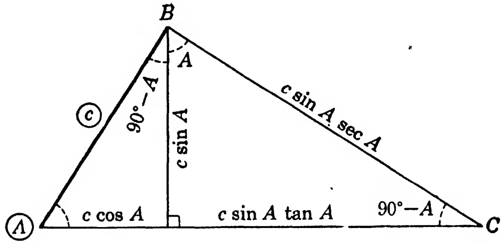


FIG. 3-17.

Pythagorean theorem. In accordance with it,

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2. \tag{18}$$

Replacing the lengths of the line segments in (18) by their values from Fig. 3-17, we get the identity

$$c^2 + c^2 \sin^2 A \sec^2 A = (c \cos A + c \sin A \tan A)^2.$$

That this is an identity may be verified in the usual way.

The student will find the following statement helpful while he is becoming familiar with the method.

To find the lengths of line segments of a rectilinear figure in terms of specified parts and to obtain identities:

(*a*) *Draw a figure, encircle each symbol representing a specified part, and put a letter on each of the other parts.*

(*b*) *Find all angles of the figure in terms of encircled angles.*

(*c*) *Use the definitions of the trigonometric functions to express all parts in terms of specified parts.*

(*d*) *Form identities by using the definitions of the trigonometric functions, by equating two expressions for the same length or area, and by using theorems from geometry.*

EXERCISES 3-5

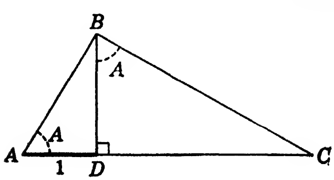


FIG. 3-18.

1. In Fig. 3-18 show that $AB = \sec A$, $BD = \tan A$, $BC = \tan A \sec A$, $DC = \tan^2 A$. Write each of these values on the appropriate line of the figure and then apply the Pythagorean theorem to triangle ABC to obtain an identity.

2. In Fig. 3-19 find DE and CE in terms of a, A, and B.

Hint. Find in order the lengths DF, DE, FE, CF, CE.

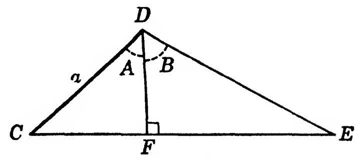


FIG. 3-19.

3. In Fig. 3-20 find the length of OE.

Hint. Find in succession the lengths OB, OC, OD, and OE.

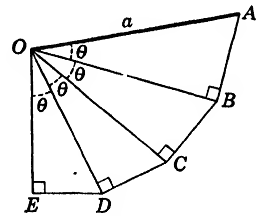


FIG. 3-20.

4. In Fig. 3-20 replace θ by $(90° - \theta)$, and then find the length of OE in the resulting figure.

5. Compute the lengths of AB and AD in Fig. 3-21.

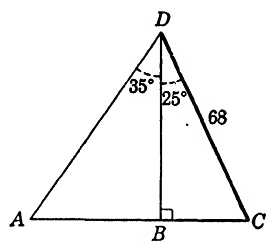


FIG. 3-21.

6. Compute lengths FE and BC in Fig. 3-22 (angle $ABE \neq 90°$).

Hint. To find the length of BC, find in succession the lengths x, y, BC.

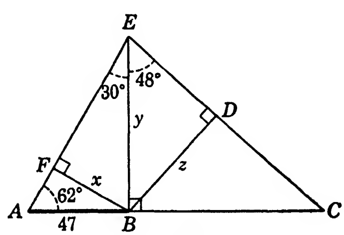


FIG. 3-22.

7. In Fig. 3-23 find the lengths DC, BC, and AB, and then read from the figure a formula for $\tan \frac{1}{2}\theta$ in terms of $\sin \theta$ and $\cos \theta$.

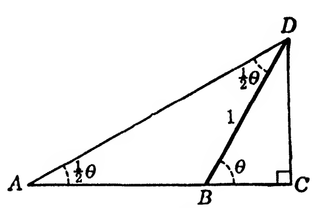


FIG. 3-23.

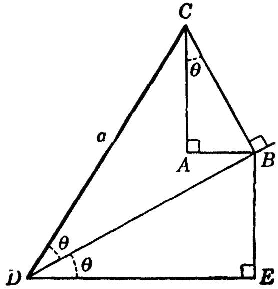


FIG. 3-24.

8. In Fig. 3-24 AB is parallel to DE. Find AB and DE in terms of a and θ.

Hint. Find in succession the lengths CB, AB, DB, DE.

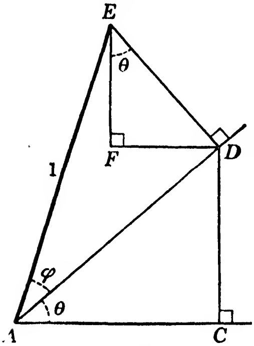


FIG. 3-25.

9. In Fig. 3-25 find in succession the lengths ED, FE, FD AD, CD, AC in terms of θ and φ, and write each of them on the appropriate line segment of the figure.

10. In Fig. 3-25 erase 1 from AE, take $AC = 1$, and find in succession the lengths CD, AD, DE, FE, FD.

11. Draw an isosceles triangle with vertical angle equal to 2θ; drop a perpendicular from the vertical angle to the side opposite and a perpendicular from a second angle to the side opposite. Find the values of all line segments in the figure thus drawn. Write two expressions for the area of the triangle and equate them to obtain an identity.

MISCELLANEOUS EXERCISES 3-6

1. Express as trigonometric functions of angles less than 45°:

(*a*) $\sin 65°$. (*b*) $\tan 49°$. (*c*) $\sec 82°$.

2. Simplify:

(*a*) $\cot\theta \tan(90° - \theta)\sin^2\theta$.
(*b*) $\sin\theta \tan\theta \cos\theta + \cos^2\theta$.
(*c*) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$.
(*d*) $\sin\theta \csc\theta + \tan^2\theta$.
(*e*) $\left(\dfrac{\sin\theta}{\cos\theta}\right)^2 + \sec\theta\cos\theta$.
(*f*) $\cot(90° - \theta)\sin\theta\cos\theta$.
(*g*) $\cot(90° - A) - \tan A + \sin 90° + \tan 45°$.

3. Transform each of the expressions in the left-hand column into the one written to the right of it.

(*a*)	$\sin\theta \cot\theta.$	$\cos\theta.$
(*b*)	$\sin\theta \sec\theta.$	$\tan\theta.$
(*c*)	$\dfrac{\cos^2 A}{1 - \sin A}.$	$1 + \sin A.$
(*d*)	$\dfrac{\csc^2\theta - 1}{\sec^2\theta - 1}.$	$\cot^4\theta.$
(*e*)	$\dfrac{1}{\sec A - \cos A}.$	$\cot A \csc A.$
(*f*)	$\dfrac{1 + \sin A}{1 - \sin A} - \dfrac{1 - \sin A}{1 + \sin A}.$	$4 \tan A \sec A.$
(*g*)	$\csc^4 A - \cot^4 A.$	$\csc^2 A - \cot^2 A.$
(*h*)	$\cos\theta\sqrt{\sec^2\theta - 1}.$	$\sin\theta.$
(*i*)	$\dfrac{1 + \sin^2 A \sec^2 A}{1 + \cos^2 A \csc^2 A}.$	$\tan^2 A.$
(*j*)	$\dfrac{1 - 2\cos^2 A}{\sin A \cos A}.$	$\tan A - \cot A.$
(*k*)	$\dfrac{1 + \cos A}{\sec A - \tan A} - \dfrac{1 - \cos A}{\sec A + \tan A}.$	$2(1 + \tan\theta).$

4. Express each of the following in terms of $\sin A$:

(*a*) $\cos A \cot A.$ (*b*) $\sin A(\cot^2) A + 1).$
(*c*) $\tan A/\sec A.$ (*d*) $\cos^4 A - \sin^4 A.$

5. Express each of the following in terms of $\cos A$:

(*a*) $\sin A \cot A.$ (*b*) $\cot^2 A/(1 + \cot^2 A).$

6. Express each of the following in terms of $\tan\theta$:

(*a*) $(\sec^2\theta - 1)\cot\theta.$ (*b*) $\sec^4\theta - \sec^2\theta.$

7. Change each of the following to equivalent forms involving only $\sin\theta$ and $\cos\theta$:

(*a*) $\tan\theta + \cot\theta.$ (*b*) $\csc\theta - \cot\theta.$ (*c*) $\sec\theta + \tan\theta.$

8. (*a*) If $x = a\cos\theta$ and $y = b\sin\theta$, show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

(*b*) If $x = a\sec\theta$ and $y = b\tan\theta$, show that $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

(*c*) If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, show that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

9. In each of the expressions in the left-hand column replace x by its value written opposite, and solve the result for y:

(a) $x^2 + y^2 = a^2.$	$x = a \cos \theta.$
(b) $b^2x^2 + a^2y^2 = a^2b^2.$	$x = a \cos \theta.$
(c) $b^2x^2 - a^2y^2 = a^2b^2.$	$x = a \sec \theta.$
(d) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$	$x = a \cos^4 \theta.$
(e) $x^{\frac{1}{3}} + y^{\frac{1}{3}} = a^{\frac{1}{3}}.$	$x = a \cos^6 \theta.$
(f) $x^2y^2 = b^2x^2 + a^2y^2.$	$x = a \sec \theta.$
(g) $x^2y^2 = a^2y^2 - b^2x^2.$	$x = a \sin \theta.$
(h) $y^2(2a - x) = x^3.$	$x = 2a \sin^2 \theta.$
(i) $y^2(x^2 + 4a^2) = 16a^4.$	$x = 2a \tan \theta.$

Verify the identities numbered 10 to 30.

10. $\sec x - \cos x = \sin x \tan x.$

11. $\tan^2 x \csc^2 x \cot^2 x \sin^2 x = 1.$

12. $\tan^2 x \cos^2 x + \sin^2 x \cot^2 x = 1.$

13. $(1 + \tan \theta)(1 + \cot \theta) \sin \theta \cos \theta = 1 + 2 \sin \theta \cos \theta.$

14. $(\tan \theta + \cot \theta)^2 = \sec^2 \theta \csc^2 \theta.$

15. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x.$

16. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x.$

17. $\sin \theta \cos \theta (\sec \theta + \csc \theta) = \sin \theta + \cos \theta.$

18. $\sin^2 x \sec^2 x = \sec^2 x - 1.$

19. $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}.$

20. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1.$

21. $\cot A + \dfrac{\sin A}{1 + \cos A} = \dfrac{1}{\sin A}.$

22. $\sec^4 \theta - 1 = 2 \tan^2 \theta + \tan^4 \theta.$

23. $\dfrac{\csc \theta}{\cot \theta + \tan \theta} = \cos \theta.$

24. $(\tan \theta + \sec \theta)^2 = \left(\dfrac{1 + \sin \theta}{\cos \theta}\right)^2.$

25. $\sin x(1 + \tan x) + \cos x(1 + \cot x) = \sec x + \csc x.$

26. $\dfrac{\sin x}{1 + \cos x} + \dfrac{1 + \cos x}{\sin x} = 2 \csc x.$

27. $\dfrac{\cos \theta}{1 - \tan \theta} + \dfrac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta.$

28. $\dfrac{1 - \cos \theta}{1 + \cos \theta} = \dfrac{(1 - \cos \theta)^2}{\sin^2 \theta}.$

29. $\dfrac{\sec x}{1 + \cos x} = \dfrac{\tan x - \sin x}{\sin x(1 - \cos^2 x)}.$

30. $\cot x + \csc x = \dfrac{\sin x}{1 - \cos x}.$

31. In Fig. 3-26 compute the length of x.

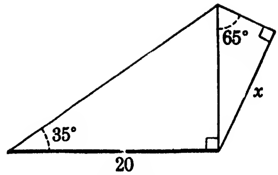


FIG. 3-26.

32. Compute the lengths of AB and AD in Fig. 3-27.

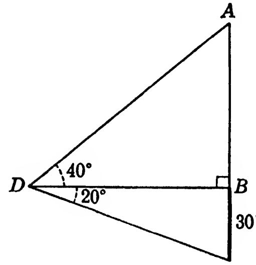


FIG. 3-27.

33. Compute the length of each line segment in Fig. 3-28.

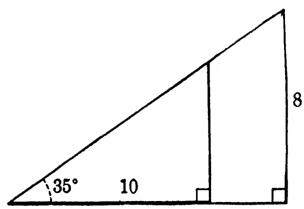


FIG. 3-28.

34. In Fig. 3-29 compute y by first finding x.

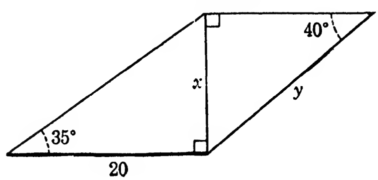


FIG. 3-29.

35. In Fig. 3-30 find the lengths of AC and AB in terms of a, θ, ϕ, and α.

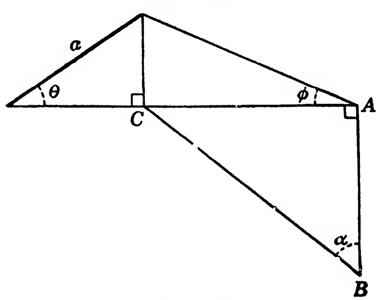


FIG. 3-30.

CHAPTER 4

GENERAL DEFINITIONS OF TRIGONOMETRIC FUNCTIONS

4-1. Definition of angle. Only trigonometric functions of angles no greater than 90° have been considered in the preceding chapters. This chapter will be concerned with functions of angles that may have any magnitude.

A half line or ray is the part of a straight line lying on one side of a point of the line. It is designated by naming its end point and another point on it. Thus OA in Fig. 4-1 is the ray beginning at O and extending through A. If a half line or ray beginning at point O rotates about O in a plane from an initial position OA to a terminal position OB, it is said to generate the angle AOB (see Fig. 4-1). When the legs of a compass are drawn apart an angle is generated; the hands of a clock rotate and generate angles.

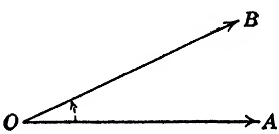
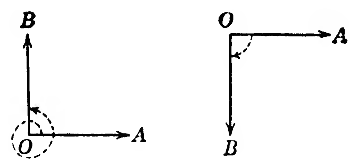


FIG. 4-1.

When the generating ray is turned through one-fourth of the complete turn about a point, the angle generated is called a right angle; a degree is $\frac{1}{90}$ of a right angle, a minute is $\frac{1}{60}$ of a degree, and a second is $\frac{1}{60}$ of a minute. Although either direction of rotation may be considered positive, it is customary in trigonometry to call angles generated by **counter-clockwise** rotation **positive** angles and those generated by **clockwise** rotation **negative** angles. In Fig. 4-2(a) the curved arrow indicates counterclockwise or positive rotation through five right angles; in Fig. 4-2(b) a negative right angle is indicated.



Counter clockwise or positive rotation
(a)

Clockwise or negative rotation
(b)

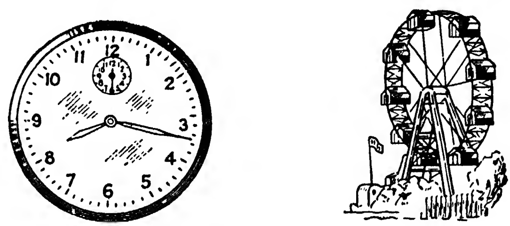
FIG. 4-2.

EXERCISES 4-1

1. Construct the following angles:

(a) 6 right angles. (b) -6 right angles.
(c) 5 right angles. (d) -3 right angles.
(e) $3\frac{1}{3}$ right angles. (f) $-2\frac{1}{2}$ right angles.

2. Through how many right angles does the minute hand of a clock turn from 12:15 P.M. to 2 P.M. of the same day [see Fig. 4-3(a)]?



(a) (b)
FIG. 4-3.

3. What are the magnitude and sense of the angles generated by the hour hand of a clock between 3 A.M. and the next 8 A.M.?

4. Through what part of a right angle does the minute hand of a clock move in 1 min. of time?

5. A Ferris wheel is turning through 3 revolutions in each minute. Through how many right angles will it turn in 2 min. [see Fig. 4-3(b)]?

6. An imaginary line connecting the center of the earth's orbit to the center of the earth makes one complete revolution each year. Assuming that this line turns in a plane at a constant rate, find the number of right angles described by this line in (a) 3 months; (b) 7 months; (c) 25 months; (d) 2000 years; (e) 1 day; (f) 1 hr.

4-2. Rectangular coordinates. This article is designed to recall the essential conceptions of rectangular coordinates, which are used in the definitions of the trigonometric functions of any angle.

In Fig. 4-4, $X'X$ represents a straight line, and O is any point on it. If we choose a unit of measure, any point to the right of O will be designated by a positive number telling its distance from O in terms of the chosen unit, and any point to the left of O will

be designated by a negative number whose magnitude gives the distance of the point from O. Thus a point 5 units to the right of O is designated by 5, whereas a point $3\frac{1}{2}$ units to the left of O is designated by -3.5.

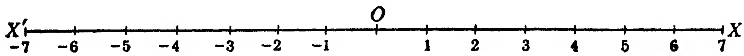


FIG. 4-4.

By means of a system called rectangular coordinates, the position of any point in the plane is defined by two numbers. In this system two mutually perpendicular lines, referred to as **axes,** are required. In Fig. 4-5, $X'X$ and $Y'Y$ represent two perpendicular lines intersecting at O. The four parts into which the plane is divided by these lines are called the first, second, third, and fourth quadrants, respectively, as indicated in the figure. Let P be any point in the plane of $X'X$ and $Y'Y$. Drop a perpendicular from P to the x-axis, meeting it in Q, and another from P to the y-axis, meeting it in R. Let x, considered as positive when P is to the right of $Y'Y$ and as negative when P is to the left of $Y'Y$, be the measure of OQ in terms of a given unit of measure; let y, considered as positive when P is above $X'X$ and negative when P is below $X'X$, be the measure of OR in terms of the given unit. Then any point in the plane will be represented by a pair of numbers, x and y.

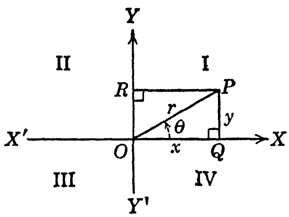


FIG. 4-5.

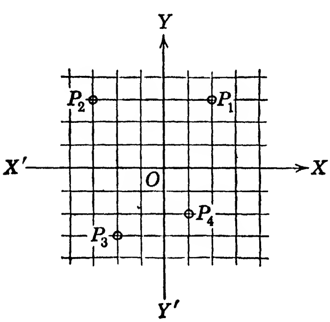


FIG. 4-6.

The first number x is called the **abscissa** of the point P, and the second number y is called its **ordinate.** The two numbers x and y are called the **coordinates** of P, and the point is designated (x, y). Thus in Fig. 4-6 the abscissa of P_1 is 2, its ordinate is 3, its coordinates are 2 and 3, and it is designated (2, 3). Similarly, P_2 is designated $(-3, 3)$, P_3 is designated $(-2, -3)$, and P_4 is designated $(1, -2)$.

EXERCISES 4-2

1. Plot the points (2, 4), (−2, 4), (2, −4), (−2, −4), (4, 2), (4, −2), (−4, 2), (−4, −2). Why do all these points lie on a circle?

2. Plot the points (0, 1), (0, 5), (1, 0), (5, 0), (0, −1), (0, −5), (−1, 0), (−5, 0), (0, 0).

3. Read the trigonometric functions of the angle subtended at O by the line connecting (*a*) (12, 0) to (12, 5); (*b*) $(x, 0)$ to (x, y), assuming x and y to be positive numbers.

4. Where are all the points for which (*a*) $x = 3$? (*b*) $y = -3$? (*c*) $x = -4$? (*d*) $y = 5$? (*e*) $x = 0$? (*f*) $y = 0$? (*g*) $r = 3$?

5. What is the abscissa of all points on the y-axis? What is the ordinate of all points on the x-axis?

6. Determine the quadrant in which (*a*) the abscissa and ordinate are both positive; (*b*) the abscissa is negative and the ordinate is positive; (*c*) the abscissa is positive and the ordinate is negative; (*d*) the abscissa and ordinate are both negative.

7. Assuming that r is always positive, in which quadrants are each of the following ratios positive? in which negative?

(*a*) $\dfrac{y}{r}$. (*b*) $\dfrac{x}{r}$. (*c*) $\dfrac{x}{y}$. (*d*) $\dfrac{y}{x}$. (*e*) $\dfrac{r}{x}$. (*f*) $\dfrac{r}{y}$.

4-3. Definitions of the trigonometric functions of any angle. In considering the functions of an angle in any quadrant, study

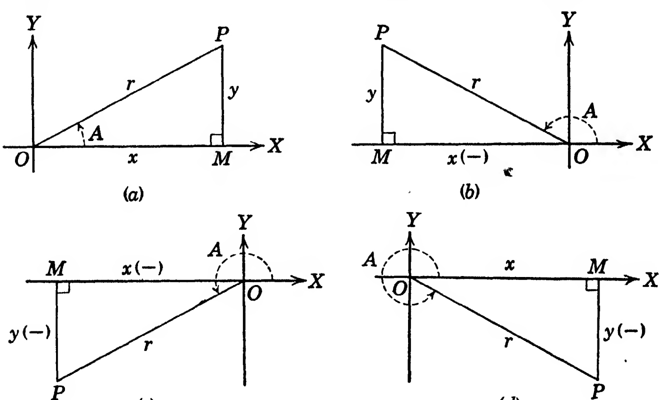


FIG. 4-7.

the four diagrams in Fig. 4-7. In (*a*) you see an acute angle in the first quadrant. We shall call PM, the side opposite angle A, the ordinate, and OM, the side adjacent to angle A, the abscissa.

We shall call the hypotenuse, OP, the distance. The definitions of the six trigonometric functions in Arts. 1-3 and 1-5 will now be changed to the following:

$$\left.\begin{aligned}
\sin A &= \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r}, & \csc A &= \frac{\text{distance}}{\text{ordinate}} = \frac{r}{y},\\
\cos A &= \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}, & \sec A &= \frac{\text{distance}}{\text{abscissa}} = \frac{r}{x},\\
\tan A &= \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}, & \cot A &= \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}.
\end{aligned}\right\} \qquad (1)$$

As angle A increases and appears in the second, third, and fourth quadrants, PM is still the ordinate, OM is still the abscissa, and OP is still the distance. The new definitions hold also for the angles in (*b*), (*c*), and (*d*), that is, for any angle in any quadrant. It is important to keep in mind that in (*b*) and (*c*) OM is negative, and that in (*c*) and (*d*) PM is negative. Thus, in Fig. 4-8 and in Fig. 4-9

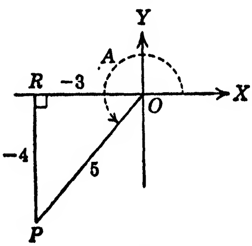


FIG. 4-8.

$$\sin A = \frac{-4}{5}, \qquad \csc A = \frac{5}{-4},$$
$$\cos A = \frac{-3}{5}, \qquad \sec A = \frac{5}{-3},$$
$$\tan A = \frac{-4}{-3} = \frac{4}{3}, \qquad \cot A = \frac{-3}{-4} = \frac{3}{4}.$$

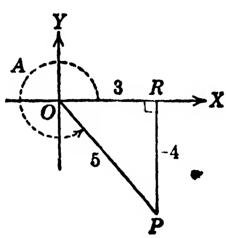


FIG. 4-9.

$$\sin A = \frac{-4}{5}, \qquad \csc A = \frac{5}{-4},$$
$$\cos A = \frac{3}{5}, \qquad \sec A = \frac{5}{3},$$
$$\tan A = \frac{-4}{3}, \qquad \cot A = \frac{3}{-4}.$$

EXERCISES 4-3

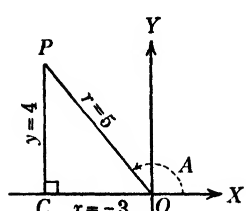


FIG. 4-10.

1. Read the values of the trigonometric functions of an angle A if its cosine is $-\frac{3}{5}$ and (*a*) if it is a second-quadrant angle (see Fig. 4-10); (*b*) if it is a third-quadrant angle.

2. Write the appropriate signs, + or −, in the blank spaces of the following form:

	sin	cos	tan	cot	sec	csc
1st quad						
2d quad						
3d quad						
4th quad						

3. The sine of a certain angle is $-\frac{1}{2}$, and its cosine is $\frac{\sqrt{3}}{2}$. Find the values of the other trigonometric functions of this angle.

4. Fill in the blank spaces of the following diagram:

Angle	sin	cos	tan	cot	sec	csc
A	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$				2
A			1		$-\sqrt{2}$	
A				$-\sqrt{3}$		-2
A	$\frac{5}{13}$	$-\frac{12}{13}$				

5. The absolute value (numerical value without reference to sign) of the tangent of an angle is $\frac{5}{12}$. Write the values of the six trigonometric functions of this angle (*a*) when it is less than 90°; (*b*) when it is greater than 90° but less than 180°; (*c*) when it is greater than 180° but less than 270°; (*d*) when it is greater than 270° but less than 360°.

6. Each of the following points is on the terminal side of an angle θ in standard position; find the trigonometric functions of θ.

(*a*) (4, 3). (*b*) (−4, 3). (*c*) (−5, −12).
(*d*) (15, −8). (*e*) (24, −7). (*f*) (1, 3).
(*g*) (2, −3). (*h*) $(1, \sqrt{3})$. (*i*) (−2, 0).

7. In what quadrants may θ terminate under the following conditions:

(*a*) $\sin\theta$ pos.? (*b*) $\cos\theta$ neg.? (*c*) $\tan\theta$ pos.?
(*d*) $\cot\theta$ neg.? (*e*) $\sec\theta$ neg.? (*f*) $\csc\theta$ pos.?

8. In what quadrant must θ terminate under the following conditions:

(*a*) $\sin\theta$ pos. and $\cos\theta$ neg.? (*b*) $\tan\theta$ neg. and $\sec\theta$ pos.?
(*c*) $\cot\theta$ neg. and $\cos\theta$ pos.? (*d*) $\cos\theta$ neg. and $\sin\theta$ neg.?
(*e*) $\cos\theta$ neg. and $\csc\theta$ pos.? (*f*) $\cot\theta$ neg. and $\csc\theta$ neg.?

9. Locate the terminal side of θ and find its other functions, having given:

(*a*) $\cos\theta = \frac{4}{5}$, $\sin\theta$ pos. (*b*) $\tan\theta = -\frac{12}{5}$, $\sin\theta$ neg.
(*c*) $\sin\theta = -\frac{8}{17}$, $\cot\theta$ neg. (*d*) $\sec\theta = \frac{4}{3}$, $\tan\theta$ neg.
(*e*) $\csc\theta = -\frac{17}{8}$, $\tan\theta$ pos. (*f*) $\cot\theta = -\frac{8}{15}$, $\csc\theta$ neg.
(*g*) $\sin\theta = \frac{1}{2}$, $\cos\theta$ neg. (*h*) $\sec\theta = -2$, $\sin\theta$ neg.
(*i*) $\tan\theta = -\frac{5}{12}$, $\sec\theta$ pos. (*j*) $\cot\theta = -\frac{4}{3}$, $\sin\theta$ neg.
(*k*) $\cos\theta = \frac{5}{13}$, $\cot\theta$ neg. (*l*) $\csc\theta = -2$, $\tan\theta$ neg.

10. Find the value of $2\tan\theta/(1 - \tan^2\theta)$ when $\cos\theta = -\frac{3}{5}$ and θ is in the third quadrant.

11. Find the value of $(\csc\theta - \cot\theta)(\sin^2\theta + \cos^2\theta)$ when

$$\sec\theta = -\frac{5}{4}$$

and $\tan\theta$ is negative.

12. If $\sin\theta = \frac{3}{5}$, find the values of $(\cos\theta - \csc\theta)/\cot\theta$ for the various quadrants in which θ may terminate.

4-4. Observations. We have seen in Arts. 1-4 and 1-5 that each of the six trigonometric functions of an acute angle has only one value. Similarly, each of the trigonometric functions of an angle, unrestricted in magnitude, has only one value. However, the converse is not true. Since the trigonometric functions are defined in terms of values dependent on an initial ray and a terminal ray, each of them has the same value for a given angle as for any other angle having the same initial position and the same terminal position as the given angle. In other words, *the value of any trigonometric function of a given angle is equal to the value of the same trigonometric function of any angle differing from the given one by a multiple of* 360°. Hence, in finding the value of a trigonometric function of any angle, one may add to the angle or subtract from it any integral multiple of 360°.

Observing that x is negative and that y and r are positive in the second quadrant, we see that the sin θ (y/r) and csc θ (r/y) are positive and the other four trigonometric functions are negative for second quadrant angles. Similarly, x and y are both negative in the third quadrant, so that the tangent (y/x) and the cotangent (x/y) are both positive, and the other functions are negative for third quadrant angles. Finally, in the fourth quadrant, x and r are positive, so that the cosine (x/r) and the secant (r/x) are positive and the other functions are negative for fourth quadrant angles.

4-5. Values of trigonometric functions for special angles. In Arts. 1-6 and 1-7 we were able to read from appropriate figures the trigonometric functions of 0°, 30°, 45°, 60°, and 90°. Now we are able to consider the values of the trigonometric functions of related angles in other quadrants.

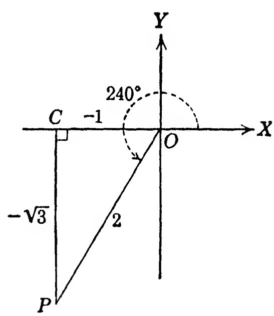


FIG. 4-11.

For example, to find the trigonometric functions of 240°, draw the line OP (Fig. 4-11) so that angle XOP is 240°. Therefore, angle $COP = 240° - 180° = 60°$. Take the distance OP as 2 units, draw PC perpendicular to the x-axis, and compute $OC = -1$ and $CP = -\sqrt{3}$. We then read from the diagram

$$\sin 240° = -\frac{\sqrt{3}}{2}, \qquad \csc 240° = -\frac{2}{\sqrt{3}},$$

$$\cos 240° = -\tfrac{1}{2}, \qquad \sec 240° = -2,$$

$$\tan 240° = \sqrt{3}, \qquad \cot 240° = \frac{1}{\sqrt{3}}.$$

Likewise, from Fig. 4-12 we read

$$\sin 135° = \frac{1}{\sqrt{2}}, \qquad \sec 135° = -\sqrt{2},$$

$$\cos 135° = -\frac{1}{\sqrt{2}}, \qquad \csc 135° = \sqrt{2},$$

$$\tan 135° = -1, \qquad \cot 135° = -1.$$

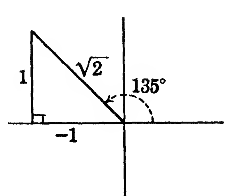


FIG. 4-12.

By a method similar to that used in Art. 1-7, we read from Figs. 4-13(a) and (b) the values of the trigonometric functions

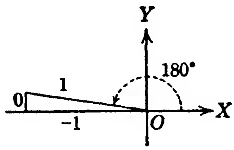


FIG. 4-13(a).

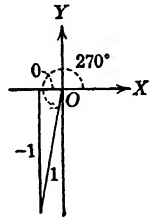


FIG. 4-13(b).

of 180° and 270°, tabulated below.

TABLE A

Angle	sin	cos	tan	cot	sec	csc
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1

EXERCISES 4-4

1. Draw a figure similar to Fig. 4-11 but designed for an angle of 210°. From this figure read the values of the trigonometric functions of 210°.

2. Make a tabular form, similar to that of Table A above, containing a blank space for each of the values of the six trigonometric functions of 0°, 60°, 90°, 120°, 135°, $-135°$, 270°, $-60°$, 315°. Then fill in the blank spaces of the form from figures prepared for the purpose.

3. Find two positive angles A less than 360° for which

(a) $\sin A = \frac{1}{2}$. (b) $\sin A = -\frac{1}{2}$.
(c) $\tan A = \frac{1}{3}\sqrt{3}$. ($d$) $\tan A = -\frac{1}{3}\sqrt{3}$.
(e) $\cos A = 1/\sqrt{2}$. (f) $\sec A = -\sqrt{2}$.

4. Find all positive angles less than 360° for which

(a) $\sin A = 1$. (b) $\cos A = -1$. (c) $\tan A = 0$.
(d) $\cos A = 0$. (e) $\sin A = 0$. (f) $\csc A = -1$.
(g) $\cot A = 0$. (h) $\tan A = \infty$. (i) $\cot A = \infty$.

5. Find the values of the trigonometric functions of (a) 165°; (b) 285°; (c) 245°; (d) 205°; (e) 105°.

Hint. Use the table in Art. 2-1.

6. Evaluate $4\sqrt{3}\tan 150° + 3\sin 90°\tan 225° - 6\sin 330° + \cos 270°$.

7. Evaluate (*a*) $\sin 60° - 2\sin 330°$; (*b*) $2\sin 45° - \sin 690°$; (*c*) $3\cos 60° - \cos 180°$; (*d*) $3\sin 690° - \sin 90°$.

8. Evaluate $4\sin 90°\sin 330°\sin 180° + (1/\sqrt{3})\tan 240°$.

9. Show that $\sin 120° = \sin 180°\cos 60° - \cos 180°\sin 60°$.

10. Show that

$$\tan 210° = \frac{\tan 240° - \tan 30°}{1 + \tan 240°\tan 30°}.$$

11. Show that

$$\cot 330° = \frac{\cos 120°\cos 210° - \sin 120°\sin 210°}{\sin 120°\cos 210° + \cos 120°\sin 210°}.$$

12. Verify that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

for each of the following values of θ: (*a*) $\theta = 45°$; (*b*) $\theta = 135°$; (*c*) $\theta = 120°$.

13. Verify that $\sin 4\theta = 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$ for each of the following values of θ: (*a*) $\theta = 30°$; (*b*) $\theta = 120°$; (*c*) $\theta = 210°$.

14. Verify that $\sin(A + B) = \sin A\cos B + \cos A\sin B$ for (*a*) $A = 210°$, $B = 30°$; (*b*) $A = 135°$, $B = 225°$.

15. Verify that $\cos(A + B) = \cos A\cos B - \sin A\sin B$ for (*a*) $A = 120°$, $B = 210°$; (*b*) $A = 315°$, $B = 135°$.

16. Evaluate:

(*a*) $\dfrac{\cos 150°\tan 300°}{\cot 225° + \sin(-30°)}$. (*b*) $\dfrac{\sec^2 135°}{\cos(-240°) - 2\sin 210°}$.

(*c*) $\dfrac{\tan^3 315°}{2\sin^2 240° + \cos 180°}$. (*d*) $\dfrac{\sin 90° - 3\cot 495°}{\cos 510°\csc(-60°)}$.

4-6. Fundamental identities. The fundamental identities developed in Chap. 3 are true for all angles. They are

$$\sin A = \frac{1}{\csc A}, \qquad \cos A = \frac{1}{\sec A}, \qquad \tan A = \frac{1}{\cot A},$$

$$\csc A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A},$$

$$\tan A = \frac{\sin A}{\cos A}, \qquad \cot A = \frac{\cos A}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1, \quad \tan^2 A + 1 = \sec^2 A, \quad \cot^2 A + 1 = \csc^2 A.$$

The argument used in Chap. 3 to prove the identities for acute angles may be extended to apply to angles of any magnitude, provided no angles are considered for which any function involved is undefined. This may be done by replacing a by x, b by y, and c by r in those arguments. Later in the chapter, it will be shown that the complementary relations developed in Chap. 3, namely,

$$\cos(90° - A) = \sin A, \qquad \sin(90° - A) = \cos A,$$
$$\cot(90° - A) = \tan A, \qquad \tan(90° - A) = \cot A,$$
$$\csc(90° - A) = \sec A, \qquad \sec(90° - A) = \csc A,$$

are true also for all values of angle A. Since only permissible algebraic operations and the identities just referred to were used in the verifications in Chap. 3, all these verifications apply whether the angle is acute or not.

4-7. Expressing a trigonometric function of any angle as a function of an acute angle. By using the generalized definitions of the trigonometric functions (Art. 4-3), it is possible to express any one of the six functions of an angle as plus or minus a trigonometric function of a positive angle less than 90°. In fact, they can be expressed as functions of an angle no greater than 45°. Consider, for example, the problem of expressing the six functions of 230° in terms of an angle less than 90°. In Fig. 4-14, angle $XOP = 230°$. The coordinates of P are $-x$ and $-y$. PO is prolonged into the first quadrant so that $OP_1 = OP = r$. Triangle OP_1R_1 is congruent to triangle OPR; and the coordinates of P_1 are x and y. Hence, $\sin 230° = \frac{-y}{r} = -\frac{y}{r}\cdot$ But, from triangle R_1OP_1, $\sin 50° = \frac{y}{r}\cdot$ Since $-\frac{y}{r} = -\left(\frac{y}{r}\right)$,

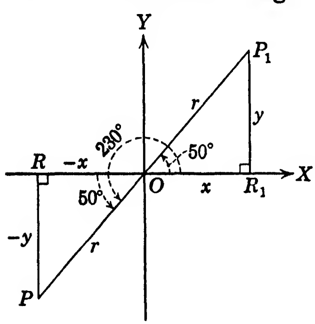


FIG. 4-14.

$$\sin 230° = -\sin 50°.$$

Likewise, $\cos 230° = \frac{-x}{r} = -\frac{x}{r} = -\left(\frac{x}{r}\right) = -\cos 50°$,

$$\tan 230° = \frac{-y}{-x} = \frac{y}{x} = \tan 50°,$$

$$\cot 230° = \frac{-x}{-y} = \frac{x}{y} = \cot 50°,$$

$$\sec 230° = \frac{r}{-x} = -\frac{r}{x} = -\left(\frac{r}{x}\right) = -\sec 50°,$$

$$\csc 230° = \frac{r}{-y} = -\frac{r}{y} = -\left(\frac{r}{y}\right) = -\csc 50°.$$

Since for acute angle θ any function of θ = co-function $(90° - \theta)$, we have

$$\begin{aligned}
\sin 230° &= -\sin 50° = -\cos 40°,\\
\cos 230° &= -\cos 50° = -\sin 40°,\\
\tan 230° &= \tan 50° = \cot 40°,\\
\cot 230° &= \cot 50° = \tan 40°,\\
\sec 230° &= -\sec 50° = -\csc 40°,\\
\csc 230° &= -\csc 50° = -\sec 40°.
\end{aligned}$$

Hence, the functions of 230° are expressed as functions of 40°, an angle less than 45°.

Similarly, from Fig. 4-15, we express the functions of $-20°$ in terms of functions of 20°:

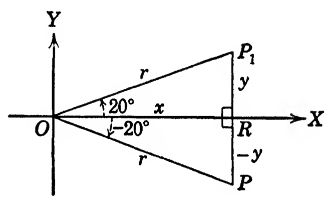


FIG. 4-15.

$$\sin(-20°) = \frac{-y}{r} = -\frac{y}{r} = -\sin 20°,$$

$$\cos(-20°) = \frac{x}{r} = \cos 20°,$$

$$\tan(-20°) = \frac{-y}{x} = -\frac{y}{x} = -\tan 20°,$$

$$\cot(-20°) = \frac{-x}{y} = -\frac{x}{y} = -\cot 20°,$$

$$\sec(-20°) = \frac{r}{x} = \sec 20°,$$

$$\csc(-20°) = \frac{r}{-y} = -\frac{r}{y} = -\csc 20°.$$

It was pointed out in Art. 4-4 that the values of the six trigonometric functions of $n360° + A$ are, respectively, identical with

those of A, provided n is an integer, positive or negative. Hence, to deal with an angle of 970°, for example, subtract 2(360°) or 720° to obtain 250° and then operate as we did with 230° above. Likewise, to deal with an angle of $-390°$, add 1(360°) to get $-30°$ and operate as we did with $-20°$ above.

EXERCISES 4-5

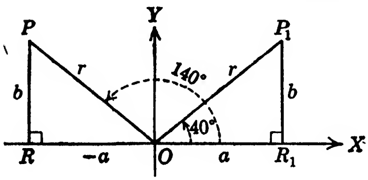


FIG. 4-16.

1. In Fig. 4-16, $OP = OP_1$. Use it to express the six trigonometric functions of 140° in terms of functions of 40°.

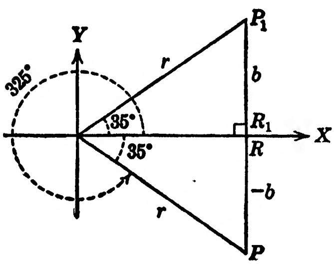


FIG. 4-17.

2. Use Fig. 4-17 to express the trigonometric functions of 325° in terms of functions of 35°.

3. Express the trigonometric functions of each of the following angles in terms of functions of an acute angle:

(*a*) 243°.	(*b*) 326°.	(*c*) 198°.
(*d*) 170°.	(*e*) 310°.	(*f*) 155°.
(*g*) 350°.	(*h*) 470°.	(*i*) 545°.
(*j*) 730°.	(*k*) $-200°$.	(*l*) 99°.
(*m*) 260°.	(*n*) 130°.	(*o*) 925°.

4-8. Functions of $180° \pm \theta$ and $360° - \theta$. In Figs. 4-18, 4-19, and 4-20, you see, respectively, angles of $180° - \theta$, $180° + \theta$,

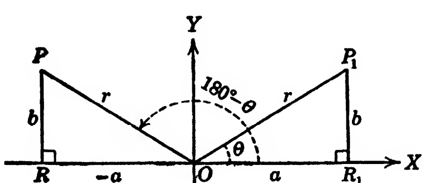


FIG. 4-18.

and $360° - \theta$. In each diagram, triangles OPR and OP_1R_1 are congruent. In the second quadrant, the coordinates of P are $-a$ and b; in the third quadrant $-a$ and $-b$; in the fourth quadrant a and $-b$. The coordinates of P_1 in each figure are a and b.

From Fig. 4-18 we have

$$\sin(180° - \theta) = \frac{b}{r} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{-a}{r} = -\left(\frac{a}{r}\right) = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{b}{-a} = -\left(\frac{b}{a}\right) = -\tan\theta,$$

$$\cot(180° - \theta) = \frac{-a}{b} = -\left(\frac{a}{b}\right) = -\cot\theta,$$

$$\sec(180° - \theta) = \frac{r}{-a} = -\left(\frac{r}{a}\right) = -\sec\theta,$$

$$\csc(180° - \theta) = \frac{r}{b} = \csc\theta.$$

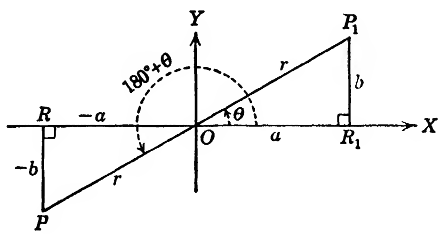


FIG. 4-19.

Similarly, from Fig. 4-19 we have

$$\sin(180° + \theta) = \frac{-b}{r} = -\sin\theta,$$

$$\cos(180° + \theta) = \frac{-a}{r} = -\cos\theta,$$

$$\tan(180° + \theta) = \frac{-b}{-a} = \tan\theta,$$

$$\cot(180° + \theta) = \frac{-a}{-b} = \cot\theta,$$

$$\sec(180° + \theta) = \frac{r}{-a} = -\sec\theta,$$

$$\csc(180° + \theta) = \frac{r}{-b} = -\csc\theta.$$

And in the same way from Fig. 4-20 we have

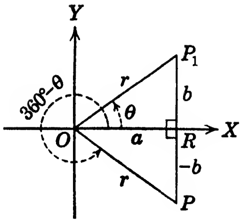


FIG. 4-20.

$$\sin(360° - \theta) = \frac{-b}{r} = -\sin\theta,$$

$$\cos(360° - \theta) = \frac{a}{r} = \cos\theta,$$

$$\tan(360° - \theta) = \frac{-b}{a} = -\tan\theta,$$

$$\cot(360° - \theta) = \frac{a}{-b} = -\cot\theta,$$

$$\sec(360° - \theta) = \frac{r}{a} = \sec\theta,$$

$$\csc(360° - \theta) = \frac{r}{-b} = -\csc\theta.$$

Thus we see that any function of $180° \pm \theta$ and of $360° - \theta$ is equal to the same function of θ. This may be written as

$$f(180° \pm \theta) = \pm f(\theta)$$

and $f(360° - \theta) = \pm f(\theta)$, in which f refers to any one of the six symbols sin, cos, tan, etc., and the plus or minus sign in the right-hand member is to be used according as the left-hand member is a positive quantity or a negative quantity. Since any integral multiple of 360° may be added to an angle, these equations could be replaced by $f(k\,180° \pm \theta) = \pm f(\theta)$ and $f(k\,360° - \theta) = \pm f(\theta)$, in which k is an integer and the plus or minus sign in the right-hand member is to be used according as the left-hand member is positive or negative.

Example. For each of the following expressions write an equivalent expression involving only an acute angle:

(*a*) cos 138°, (*b*) tan 295°, (*c*) sin 235°.

Solution. (*a*) $\cos 138° = \cos(180° - 42°) = -$ **cos 42°.** The minus sign was chosen in the right-hand member because cos 138° is negative.

(*b*) Similarly $\tan 295° = \tan(2 \times 180° - 65°) = -$ **tan 65°.** The minus sign was chosen in the right-hand member because tan 295° is a negative quantity.

(*c*) $\sin 235° = \sin(180° + 55°) = -$ **sin 55°.**

4-9. Functions of $-\theta$. In Fig. 4-21 you see an angle $-\theta$. Since any function of $-\theta$ is also the function of angle XOP, treat angle XOP as was done in Fig. 4-18 in Art. 4-8. Thus you will obtain the functions of angle $-\theta$.

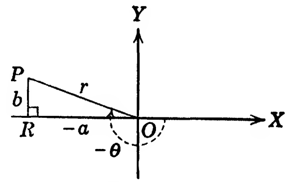


FIG. 4-21.

Example. For each of the following expressions write an equivalent expression involving a positive acute angle: (*a*) $\sin(-220°)$, (*b*) $\tan(-170°)$.

Solution. (*a*)

$$\sin(-220°) = \sin 140° = \sin(180° - 40°) = \mathbf{\sin 40°}.$$

$$(b)\ \tan(-170°) = \tan 190° = \tan(180° + 10°) = \mathbf{\tan 10°}.$$

EXERCISES 4-6

1. Use the method of this article to express the trigonometric functions of the following angles in terms of trigonometric functions of angles less than 90°: (*a*) 265°, (*b*) 275°, (*c*) 125°.

2. For each of the following expressions use the method of this article to write an equivalent one in terms of an angle no greater than 45°: $\sin 85°$, $\tan 338°$, $\sec 247°$, $\cos 197°$, $\cot 130°$, $\csc 500°$, $\sin 640°$, $\cos 1280°$, $\tan 2220°$.

3. Express as trigonometric functions of θ each of the following:

(*a*) $\sin(360° - \theta)$. (*b*) $\cos(720° - 2\theta)$.
(*c*) $\tan(180° - \theta)$. (*d*) $\sec(540° - \theta)$.
(*e*) $\csc(2 \times 180° + \theta)$. (*f*) $\sin(360° - 2\theta)$.
(*g*) $\cot(30 \times 90° + \theta)$. (*h*) $\cos(\theta - 360°)$.

4. Using trigonometric functions and positive angles less than 360°, find three expressions equal to

(*a*) $\sin 20°$. (*b*) $\cos 50°$. (*c*) $\tan 75°$.
(*d*) $\csc 87°$. (*e*) $\sec 132°$. (*f*) $\cot 247°$.
(*g*) $\sin 328°$. (*h*) $\tan 432°$. (*i*) $\cot 550°$
(*j*) $\cos 635°$. (*k*) $\sin 740°$.

5. Prove that $\sin 20° = \sin 160° = \cos 290° = -\sin 340°$.

6. Simplify:

$$(a)\ \frac{\sin 335°}{\csc 155°} + \cos 86° \cos 94°.$$

$$(b)\ \frac{\sin 200°}{\cos 20°} \tan 70° - \sec 50° \cos 130°.$$

7. Verify:

$$(a)\ \frac{\sin\theta}{\cos(180^\circ-\theta)}+\tan(360^\circ+\theta)-\sec(180^\circ+\theta)=\sec\theta.$$

$$(b)\ \frac{\cot(180^\circ+A)}{\cot(180^\circ-A)}-\frac{\sin(360^\circ-A)}{\cos(360^\circ-A)}=\tan(720^\circ+A)-1.$$

8. Prove that

$$\cos(90^\circ+A)\cos(270^\circ-A)-\sin(180^\circ-A)\sin(360^\circ-A)=2\sin^2 A.$$

MISCELLANEOUS EXERCISES 4-7

1. The tangent of a certain angle is $-\frac{2}{3}$, and its cosine is $3/\sqrt{13}$. Find all the other trigonometric functions of this angle.

2. Find all the trigonometric functions of a third-quadrant angle whose sine is $-\frac{3}{5}$.

3. Find two positive angles A less than 360° for which

(a) $\sin A = -\frac{1}{2}$,	(b) $\tan A = \sqrt{3}$,	(c) $\cot A = -1/\sqrt{2}$,
(d) $\sec A = \sqrt{2}$,	(e) $\csc A = -2$,	(f) $\cos A = -\frac{1}{2}$.

4. For each of the following expressions write an equivalent one in terms of an angle less than 90°:

(a) sin 105°.	(b) cos 170°.	(c) sec 340°.
(d) cot 242°.	(e) csc 290°.	(f) tan 184°.

5. For each of the following expressions write an equivalent one in terms of an angle no greater than 45°:

(a) sin 170°.	(b) cos 195°.	(c) cot 285°.
(d) tan 330°.	(e) sec 100°.	(f) csc 265°.

6. Find in radical form the value of each of the following:

(a) cot 120°.	(b) cos 210°.	(c) sin 240°.
(d) csc 135°.	(e) sec 225°.	(f) tan 600°.

7. Evaluate:

$$\frac{\sin 330^\circ \cos 135^\circ}{\tan 225^\circ \cos 180^\circ}+\frac{\cot 240^\circ \cos 150^\circ}{\sec 300^\circ \sin 270^\circ}.$$

8. Evaluate: $\csc^2 300° \sin 60° \tan 150° + \sec^2 210° \cot 240° \cos^2 30°$.

9. Simplify: $\cos 255° \sec 75° \sin 100° \cos 260°$.

10. Prove that $\sin 420° \cos 390° + \cos(-300°) \sin(-330°) = 1$.

11. Prove that $\cos 570° \sin 510° - \sin 330° \cos 390° = 0$.

12. Prove that $\tan y + \tan(-x) - \tan(180° - x) = \tan y$.

13. Prove that

$$\frac{\sin(180° - y)}{\sin(270° - y)} \tan(90° + y) + \csc^2(270° - y) = 1 + \sec^2 y.$$

14. Evaluate $4\sqrt{3} \tan 330° + 3 \sin 270° \cos 90° - 6 \sin(-30°)$.

15. Find in simple radical form the value of

$$\frac{\csc 225° \sec 330° \cos 690° + \tan 240° \sin 600°}{\cot 330° \sin 240° - \cos 210° \cot 120° \sin 270°}.$$

16. Show that $\sin 240° = \sin(-90°) \sin 120° - \cos 270° \cos(-60°)$.

17. Verify that $\sin 240° = 2 \sin 120° \cos 840°$.

18. Verify that $\cos 255° = \sin 45° \sin 30° - \cos 45° \cos 30°$.

19. Verify that $\sin 195° = \sin 135° \cos 60° + \cos 135° \sin 60°$.

20. Verify that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ for (*a*) $A = 330°$, $B = 60°$; (*b*) $A = 135°$, $B = 315°$.

21. Verify that $\cos(A + B) = \cos A \cos B - \sin A \sin B$ for (*a*) $A = 30°$, $B = 60°$; (*b*) $A = 240°$, $B = 330°$.

22. Verify that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

for (*a*) $A = 240°$, $B = 120°$; (*b*) $A = 315°$, $B = 225°$.

23. Verify that

$$\cos 3A = \cos 2A \cos A - \sin 2A \sin A,$$
$$\sin 3A = \sin 2A \cos A + \cos 2A \sin A,$$

for (*a*) $A = 60°$; (*b*) $A = 135°$; (*c*) $A = 600°$.

24. Verify that

$$\tan^2 x \csc^2 x \cot^2 x \sin^2 x = 1$$

for (*a*) $x = 240°$, (*b*) $x = 300°$, (*c*) $x = 480°$.

25. Verify that

$$\frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} = \tan x + \sec x$$

for (*a*) $x = 210°$, (*b*) $x = 225°$, (*c*) $x = 315°$, (*d*) $x = 330°$.

26. Verify that

$$\csc 2A = \cot A - \cot 2A$$

for (*a*) $A = 120°$, (*b*) $A = 210°$, (*c*) $A = 225°$.

27. Verify that

$$\frac{\sin(2x + y) + \sin(2x - y)}{\sin x} = 4 \cos x \cos y$$

for (*a*) $x = 120°$, $y = 60°$; (*b*) $x = 150°$, $y = 120°$.

4-10. Functions of 90° − θ. The trigonometric functions of $90° - \theta$ have been expressed in terms of θ when θ is acute. We

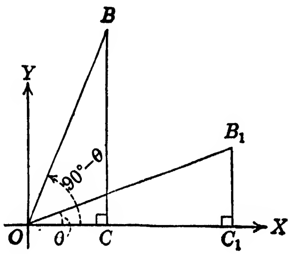


FIG. 4-22.

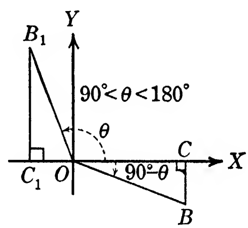


FIG. 4-23.

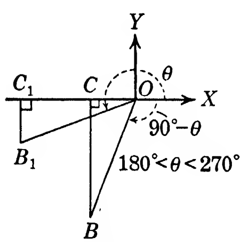


FIG 4-24.

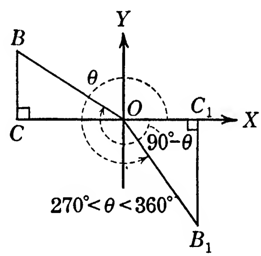


FIG. 4.25.

shall now show that these same expressions hold true when θ is any angle.

In Fig. 4-22, OX and OY represent rectangular coordinate axes and angle C_1OB_1 represents an acute angle θ. From B_1 on the terminal side of angle XOB_1, B_1C_1 is drawn perpendicular to the x-axis. Angle XOB is drawn equal to angle $90° - \theta$, OB is taken equal to OB_1, and BC is drawn perpendicular to the x-axis. In Fig. 4-23 angle θ represents an obtuse angle; in Fig. 4-24, angle θ is greater than 180° but less than 270°; and, in Fig. 4-25, angle θ is

greater than 270° but less than 360°. The description of Fig. 4-22 given above applies also to Figs. 4-23, 4-24, and 4-25 except in the statements of the magnitude of the angle θ. The two triangles OC_1B_1 and OCB in each of the four diagrams are congruent since in each case they have the hypotenuse and an acute angle of one equal, respectively, to the hypotenuse and an acute angle of the other; hence, in each figure, $OB = OB_1$, $OC = C_1B_1$, $CB = OC_1$.

Now let us agree that *a line segment MN parallel to the y-axis is positive when a point moving on this line from M to N is moving in the positive direction of the y-axis, and negative when a point moving from M to N is moving in the negative direction of the y-axis.* Thus in Fig. 4-22 the positive direction of the y-axis is toward the top of the page; hence segments C_1B_1 and CB are positive, but the same segments when read B_1C_1 and BC are considered negative. Let us agree that *a line segment MN parallel to the x-axis is positive when a point moving on this line from M to N is moving in the positive direction of the x-axis, and negative when a point moving from M to N is moving in the negative direction of the x-axis.* Thus in Fig. 4-22 the positive direction of the x-axis is to the right; hence segments OC_1 and CC_1 are positive but the same segments when read C_1O and C_1C are considered negative. Referring to Fig. 4-22, we should write $C_1O = -OC_1$, $C_1C = -CC_1$, $BC = -CB$, and $C_1B_1 = -B_1C_1$. *A line segment forming a hypotenuse will be considered positive in all cases.*

From Fig. 4-22 we read in accordance with the definitions of the trigonometric functions:

$$\sin (90° - \theta) = \frac{CB}{OB} = \frac{OC_1}{OB_1} = \cos \theta,$$

$$\cos (90° - \theta) = \frac{OC}{OB} = \frac{C_1B_1}{OB_1} = \sin \theta,$$

$$\tan (90° - \theta) = \frac{CB}{OC} = \frac{OC_1}{C_1B_1} = \cot \theta,$$

$$\cot (90° - \theta) = \frac{OC}{CB} = \frac{C_1B_1}{OC_1} = \tan \theta,$$

$$\sec (90° - \theta) = \frac{OB}{OC} = \frac{OB_1}{C_1B_1} = \csc \theta,$$

$$\csc (90° - \theta) = \frac{OB}{CB} = \frac{OB_1}{OC_1} = \sec \theta.$$

If, while reading any equation of this group, we consider the line segments involved as applying to Fig. 4-23, 4-24, or 4-25, we find that the argument holds good in each case. Moreover, the argument will still hold good in the case of each figure if angle θ represents the indicated angle increased or decreased by any number of revolutions; this is true because changing the angle θ by any number of revolutions will not change the line segments of the figure in any way. Hence, the equations are true for all values of θ.

4-11. Functions of $90° + \theta$, $270° - \theta$, and $270° + \theta$. In these cases we shall make the argument only for θ, an acute angle. However, the directions for drawing the figures and the statements made will apply for all angles θ. For each case considered below, the student may construct figures for angle θ in different quadrants, use the same letters for corresponding positions in the given figure, and note that the statements made apply to his figures as well as to the given one.

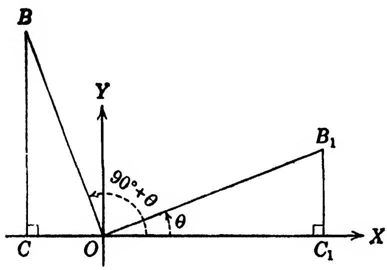


FIG. 4-26.

In Fig. 4-26, OX and OY represent rectangular axes of coordinates, angle XOB_1 represents angle θ, and angle XOB represents $90° + \theta$. B_1 is any point on the terminal side of angle θ, and B is taken on the terminal side of $90° + \theta$ so that $OB = OB_1$. The lines B_1C_1 and BC are drawn perpendicular to the x-axis and meet it in points C_1 and C, respectively. Since the triangles OB_1C_1 and OBC are congruent, $OC_1 = CB$ and $CO = C_1B_1$. Hence, we obtain

$$\sin (90° + \theta) = \frac{CB}{OB} = \frac{OC_1}{OB_1} = \cos \theta,$$

$$\cos (90° + \theta) = \frac{OC}{OB} = \frac{-C_1B_1}{OB_1} = -\sin \theta,$$

$$\tan (90° + \theta) = \frac{CB}{OC} = \frac{OC_1}{-C_1B_1} = -\cot \theta,$$

$$\cot (90° + \theta) = \frac{OC}{CB} = \frac{-C_1B_1}{OC_1} = -\tan \theta.$$

$$\sec (90° + \theta) = \frac{OB}{OC} = \frac{OB_1}{-C_1B_1} = -\csc \theta,$$

$$\csc (90° + \theta) = \frac{OB}{CB} = \frac{OB_1}{OC_1} = \sec \theta.$$

The construction of Figs. 4-27 and 4-28 is similar to that already explained. Their description will therefore be omitted.

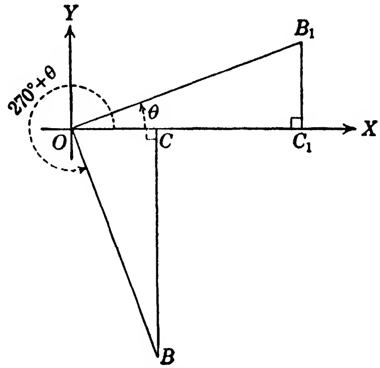


FIG. 4-27.

From Fig. 4-27 we obtain

$$\sin (270° + \theta) = \frac{CB}{OB} = \frac{-OC_1}{OB_1} = -\cos \theta,$$

$$\cos (270° + \theta) = \frac{OC}{OB} = \frac{C_1B_1}{OB_1} = \sin \theta,$$

$$\tan (270° + \theta) = \frac{CB}{OC} = \frac{-OC_1}{C_1B_1} = -\cot \theta,$$

and the other three formulas may be obtained from these by using the reciprocal relations:

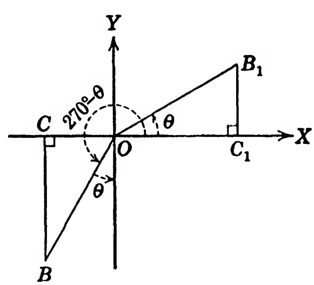


FIG. 4-28.

From Fig. 4-28 we obtain

$$\sin(270° - \theta) = \frac{-CB}{OB} = \frac{-OC_1}{OB_1} = -\cos\theta,$$
$$\cos(270° - \theta) = \frac{-CO}{OB} = \frac{-C_1B_1}{OB_1} = -\sin\theta,$$
$$\tan(270° - \theta) = \frac{-CB}{-OC} = \frac{-OC_1}{C_1B_1} = \cot\theta,$$

and the other three formulas may be obtained from these by using the reciprocal relations.

4-12. Functions of (k 90° $\pm$ θ). Observing the formulas obtained in Arts. 4-8 and 4-11, we perceive the truth of the following statements: (*a*) *each of the six trigonometric functions of* k *90°* $\pm$ θ, k *odd, is numerically equal to the co-function of* θ; (*b*) *each function of* k *90°* $\pm$ θ, k *even, is numerically equal to the same function of* θ; (*c*) *the sign to be placed before the resulting function of* θ *is the same as the sign of the original function in the quadrant of* k *90°* $\pm$ θ, *where* θ *is thought of as an acute angle.*

Although these rules are convenient, *the student will find that he can draw a rough figure and easily deduce from it the required results.*

EXERCISES 4-8

1. Draw the four figures relating to the formulas connected with $90° + \theta$. Figure 4-26 is the first figure, in the second one θ should represent an obtuse angle, in the third one θ should represent an angle greater than 180° but less than 270°, and in the fourth one θ should represent an angle greater than 270° but less than 360°. Letter your figures to correspond with Fig. 4-26 and note that the statements made in Art. 4-11 apply to each of your figures.

2. Prove formulas like those in Art. 4-11 for $270° + \theta$.

3. If the angles of a triangle are A, B, and C, express each trigonometric function of $A + B$ in terms of a function of C. Do your formulas hold true in each of the cases:

$$0° < A + B < 90°? \qquad A + B = 90°? \qquad 90° < A + B < 180°?$$

4. Using the method of Art. 4-11, express as functions of a positive angle less than 90°:

(*a*) cos 170°. (*b*) tan 110°. (*c*) cot 160°.
(*d*) sec 235°. (*e*) sin 310°. (*f*) cos 340°.

(*g*) csc 215°. (*h*) sin 100°25′. (*i*) cos 255°32′.
(*j*) tan 283°14′.

5. Express as functions of a positive angle less than 90°:

(*a*) $\cos(-20°)$. (*b*) $\tan(-80°)$. (*c*) $\sin(-120°)$.
(*d*) $\tan(-195°)$. (*e*) $\sec(-245°)$. (*f*) $\cos(-300°)$.

6. Express as functions of θ:

(*a*) $\sin(810° - \theta)$. (*b*) $\tan(360° - \theta)$.
(*c*) $\cot(270° + \theta)$. (*d*) $\sin(\theta - 90°)$.
(*e*) $\tan(\theta - 180°)$. (*f*) $\sec(-180° - \theta)$.
(*g*) $\csc(-630° + \theta)$. (*h*) $\cos(990° - \theta)$.

7. From the table of natural functions on page 327 find the sine, cosine, tangent, and cotangent of

(*a*) 100°15′. (*b*) −395°36′. (*c*) 1097°10′.
(*d*) −370°10′. (*e*) 750°53′. (*f*) −100°18′.

8. Simplify

(*a*) $\dfrac{\cos(90° + A)}{\sin(-A)} + \dfrac{\sin(90° + A)}{\cos(-A)} + \dfrac{\cot(90° + A)}{\tan(-A)}$.

(*b*) $\cos(270° - \theta)\sin(180° - \theta) - \cos(180° + \theta)\sin(270° + \theta)$.

(*c*) $\dfrac{\cos^2(180° + \theta)}{\sin^2(-\theta)} - \dfrac{\cos(270° - \theta)}{\sin(180° - \theta)}$.

(*d*) $\dfrac{\cos(180° + \theta)}{\sin(270° - \theta)} + \dfrac{\sin^3(-\theta)}{\cos(270° + \theta)}$.

(*e*) $\dfrac{\cot(270° + \theta)}{\cot(270° - \theta)} \times \dfrac{\tan(180° - \theta)}{\tan(180° + \theta)} \times \dfrac{\csc(360° - \theta)}{\sec(360° + \theta)}$.

9. Find the value of $\sin 480° \sin 690° + \cos(-420°)\cos 600°$.

10. Prove each of the following:

(*a*) $\cos 230° \cos 310° - \sin(-50°)\sin(-130°) = -1$.

(*b*) $\tan 110° \cot 340° - \sin 160° \sec 250° = \csc^2 20°$.

(*c*) $\sin(90° + \theta)\sec(270° - \theta) = \tan(270° + \theta)$.

(*d*) $\dfrac{\cos(270° + \theta)}{1 - \cos(180° - \theta)} = \dfrac{1 - \cos(-\theta)}{\cos(90° - \theta)}$.

CHAPTER 5

THE RADIAN, THE MIL, AND GRAPHS

5-1. The radian. There is a unit of angular measurement used so frequently in higher mathematics that it is understood to be the unit of measurement when no other is specified. Its importance is due to the fact that various mathematical expressions take simpler forms in terms of this unit than in terms of any other. For this reason we consider it in trigonometry. This unit is called the **radian.**

A radian is an angle which, if placed at the center of a circle, intercepts an arc equal in length to the radius of the circle. A chord of a circle, equal in length to its radius, subtends an angle of 60° at its center. An arc on the same circle, equal in length to its radius, would intercept at its center an angle slightly less than 60°. In Fig. 5-1 O is the center of the circle. Chord AC is equal to the radii AO and CO. Therefore angle $COA = 60°$. Arc AB is equal in length to the radius. Angle AOB is a radian.

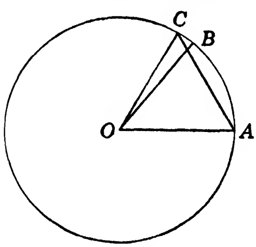


FIG. 5-1.

Since the circumference of a circle is $2\pi R$, the length of the radius is contained in the circumference 2π times. Hence, since the complete circle intercepts 360° at the center, 2π radians (that is, 6.2832 radians) are equivalent to 360°. Accordingly, we write

$$2\pi \text{ radians} = 360°, \qquad \text{or} \qquad \pi \text{ radians} = 180°. \tag{1}$$

$$\therefore\ 1 \text{ radian} = \frac{180°}{\pi} = 57.296°, \text{ or } 57.3°$$
$$= 57°17'45'', \text{ or } 57°18'. \tag{2}$$

Also, from (1), since $180° = \pi$ radians,

$$1° = \frac{\pi}{180} \text{ radian} = 0.01745 \text{ radian}. \tag{3}$$

From formulas (2) and (3) it appears that to find the number of degrees in a given number a of radians, multiply a by $180/\pi$, and to find the number of radians in a given number b of degrees, multiply b by $\pi/180$.

By way of illustration, we write

$$10^\circ = 10\left(\frac{\pi}{180}\right) \text{ radian} = \frac{\pi}{18} \text{ radian};$$

$$5' = \left(\frac{5}{60}\right)^\circ = \frac{5}{60}\,\frac{\pi}{180} \text{ radian} = \frac{\pi}{2160} \text{ radian};$$

$$0.75 \text{ radian} = 0.75\left(\frac{180}{\pi}\right)^\circ = 42.9719^\circ = 42^\circ 58' 19''.$$

EXERCISES 5-1

1. Express the following angles in radians:

(*a*) 45°. (*b*) 60°. (*c*) 90°.
(*d*) 180°. (*e*) 120°. (*f*) 135°.
(*g*) 22°30′. (*h*) 200°. (*i*) 480°.

2. Express the following angles in degrees:

(*a*) $\pi/3$ radians. (*b*) $3\pi/4$ radians. (*c*) $\pi/72$ radian.
(*d*) $7\pi/6$ radians. (*e*) $20\pi/3$ radians. (*f*) 0.98π radians.

3. Express in radians the following angles accurate to four significant figures:

(*a*) 1°. (*b*) 1′. (*c*) 3.5°.
(*d*) 10°11′. (*e*) 180°34′. (*f*) 300°25′.

4. Find, accurate to the nearest minute, the following angles in degrees and minutes: (*a*) $\frac{1}{10}$ radian; (*b*) $2\frac{1}{2}$ radians; (*c*) 1.6 radians; (*d*) 6 radians.

5. Evaluate the following (without tables):

(*a*) $\tan \frac{1}{6}\pi$. (*b*) $\sin \frac{1}{3}\pi$. (*c*) $\cos \frac{1}{4}\pi$.
(*d*) $\tan \frac{1}{3}\pi$. (*e*) $\sin \frac{1}{2}\pi$. (*f*) $\cos \pi$.
(*g*) $\cot \frac{4}{3}\pi$. (*h*) $\sec \frac{2}{3}\pi$. (*i*) $\tan (-\pi)$.

6. Find the number of radians through which each of the hands of a clock turns in (*a*) 5 min., (*b*) 15 min., (*c*) 45 min., (*d*) 2 hr., (*e*) 6 hr. 30 min.

7. Find the values of x and y in

$x = 2(\theta - \sin\theta)$ and $y = 2(1 - \cos\theta)$ when (a) $\theta = 0$, (b) $\theta = \frac{1}{3}\pi$, (c) $\theta = \frac{1}{4}\pi$, (d) $\theta = \frac{3}{4}\pi$, (e) $\theta = \frac{5}{6}\pi$, (f) $\theta = \frac{7}{6}\pi$, (g) $\theta = \frac{1}{2}\pi$, (h) $\theta = \pi$, (i) $\theta = \frac{3}{2}\pi$, (j) $\theta = 2\pi$, (k) $\theta = 7\pi$.

8. If $x = 5(\cos\theta + \theta\sin\theta)$ and $y = 5(\sin\theta - \theta\cos\theta)$, find the value of x and y when (a) $\theta = 0$, (b) $\theta = \frac{1}{3}\pi$, (c) $\theta = \frac{7}{6}\pi$.

9. Two angles of a triangle are $\frac{1}{3}\pi$ and $\frac{1}{2}$. Find the third angle in sexagesimal units. (Angle $\frac{1}{2}$ means $\frac{1}{2}$ radian.)

10. Find the numerical value of

$$(a)\ \tan\frac{11\pi}{6} - 2\sin\frac{4\pi}{3} - \frac{3}{4}\csc^2\frac{3\pi}{4} - 4\cos^2\frac{5\pi}{6}.$$

$$(b)\ \tan\frac{17\pi}{6}\tan\frac{14\pi}{3} + \cot\left(-\frac{11\pi}{6}\right)\cos\left(-\frac{4\pi}{3}\right).$$

$$(c)\ \sin\frac{19\pi}{6}\cos\left(-\frac{11\pi}{6}\right) - \sin\frac{7\pi}{3}\cos\left(-\frac{4\pi}{3}\right).$$

11. Simplify

$$\cos(\tfrac{1}{2}\pi + x)\sin(\tfrac{1}{2}\pi - x)\tan(\tfrac{3}{2}\pi - x) - \cos(\tfrac{3}{2}\pi + x)\cos(\tfrac{1}{2}\pi + x)\tan(\pi - x).$$

12. Prove

$$(a)\ \cos(\pi - x) + \tan(\pi + x)\sin(-x) = \sec(\pi + x).$$

$$(b)\ \sin\left(\frac{3\pi}{4} - \theta\right) = -\sin\left(\frac{5\pi}{4} + \theta\right).$$

$$(c)\ \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta = \cos\left(\frac{3\pi}{2} - \theta\right).$$

$$(d)\ \cos\left(\frac{\pi}{2} + x\right)\cos(\pi - x) + \sin\left(\frac{\pi}{2} + x\right)\sin(\pi + x) = 0$$

$$(e)\ \frac{\tan\pi + \tan\theta}{1 - \tan\pi\tan\theta} = \tan(\pi + \theta).$$

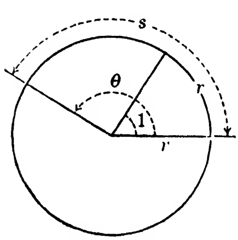


FIG. 5-2.

5-2. Length of circular arc. Figure 5-2 shows a central angle of 1 radian and a central angle of θ radians in a circle of radius r. Since two central angles in a circle have the same ratio as their intercepted arcs, we have

$$\frac{\theta}{1} = \frac{s}{r},$$

or

$$s = r\theta \text{ units.} \tag{4}$$

Example 1. A target in the form of a circular arc having its center at a gun is 3000 yd. from the gun and subtends at the gun an angle of 0.015 radian. Find the length of the target.

Solution. Here $r = 3000$ yd., and $\theta = 0.015$ radian. Substituting these numbers in (4), we obtain

$$s = r\theta = 3000(0.015) = \mathbf{45\ yd.}$$

Example 2. The nautical mile, or sea mile, used in the United States is the arc length subtended on a circle of diameter 7917.59 miles by a central angle of $1'$ (7917 miles is approximately the diameter of a sphere having a volume equal to that of the earth). Find the length of the nautical mile accurate to five figures.

Solution. Using formula (4) with

$$r = \tfrac{1}{2}(7917.6)(5280) \qquad \text{and} \qquad \theta = \tfrac{1}{60} \times \frac{\pi}{180},$$

we obtain

$$S = \tfrac{1}{2}(7917.6)(5280)\frac{\pi}{60 \times 180} = \mathbf{6080.4\ ft.}$$

This is approximately the length of the nautical mile. A more accurate value is 6080.27 ft.

EXERCISES 5-2

1. For a circle of radius 720 ft., find the length of arc subtended by a central angle of (*a*) 18°; (*b*) 28°30′; (*c*) 17°20.5′; (*d*) 40.5′; (*e*) 38′; (*f*) $(a/\pi)°$.

2. For a circle having a circumference 3000 ft. in length, find in degrees and minutes the central angle subtended by an arc of length (*a*) 300 ft.; (*b*) 10 ft.; (*c*) 1 ft.; (*d*) 12 ft.; (*e*) 2807 ft.

3. Show that a central angle of θ degrees subtends on the circumference of a circle of radius r a length s given by

$$\frac{\theta}{180} = \frac{s}{\pi r}.$$

4. If a circular arc of 30 ft. subtends 4 radians at the center of its circle, find the radius of the circle.

5. If two angles of a plane triangle are respectively equal to 1 radian and $\frac{1}{2}$ radian, express the third angle in degrees.

6. An enemy battery 6000 yd. distant from an observation post subtends at the post an angle of $\frac{1}{80}$ radian. How many yards of front does the battery occupy if the post is directly in front of it?

7. Find approximately the angle in radians subtended by a church spire of 160 ft. high at a point in the horizontal plane through the base of the spire and distant 1 mile from it.

8. An automobile whose wheels are 34 in. in diameter travels at the rate of 25 miles per hour. How many revolutions per minute does a wheel make? What is its angular velocity in radians per second?

9. Assuming the earth to be a perfect sphere 7917 miles in diameter, find the length of an arc on the equator that subtends an angle of 1° at the center of the earth. Also find the distance between two points on the same meridian if one is 8° north of the equator and the other 5°30′ south of the equator.

10. When the moon is 239,000 miles from the earth, its diameter subtends about 31′ of angle at a point on the earth. Using this fact, compute the diameter of the moon by assuming that the diameter is the arc of a circle having its center at a point on the earth.

11. The larger of two wheels about which a belt is drawn taut has a 3-ft. radius. If the centers of the wheels are 6 ft. apart and if the arc of the larger wheel in contact with the belt subtends at its center an angle of 3.4 radians, find the radius of the smaller wheel.

12. An automobile has tires 28 in. in diameter. Find the angular velocity in radians per second of the wheel of the automobile when going 50 miles per hour.

13. The drive wheel of a locomotive is 6 ft. in diameter. Find its angular velocity in radians per minute when the train is moving 60 miles per hour.

14. The drive wheel of a locomotive is 6 ft. in diameter. If it makes 500 radians per minute, find the speed of the train in miles per hour.

15. Find the average speed of a man who runs two laps in 30 sec. on a circular track that is 35 ft. in diameter.

In Exercises 16 *to* 20, *give approximate answers based on formula* (4).

16. On approaching the shore, the captain of the ship shown in Fig. 5-3 measured the angle of elevation of the top of a flagstaff and found it to be 2°10′. If he knew the height of the staff was 32 ft. and if the foot of the staff was on the same level with the captain's eye, find his distance from the flagstaff.

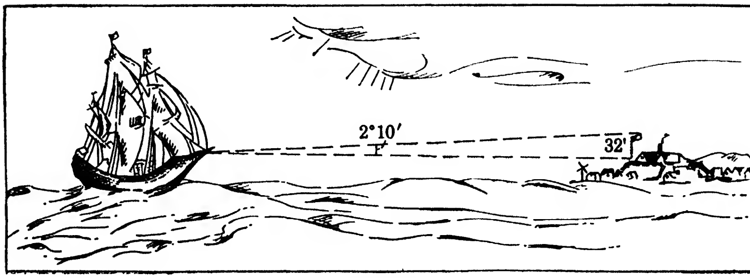


FIG. 5-3.

17. A lighthouse 100 ft. high stands on a rock. From the bottom of the lighthouse the angle of depression of a ship is 2°47′, and from the top of the lighthouse its angle of depression is 4°2′. What is the height of the rock? What is the horizontal distance from the lighthouse to the ship?

18. The signal-corps man shown in Fig. 5-4 subtends an angle of 35′ at station *S*. If he is 6 ft. tall, find his distance from the station.

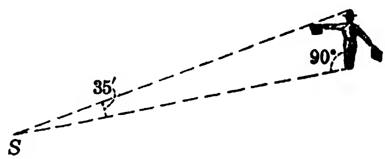


FIG. 5-4.

19. In approaching a fort situated on a plain, a reconnoitering party finds at one place that the fort subtends an angle of 3° and at a place 200 ft. nearer the fort that it subtends an angle of 6°. How high is the fort, and what is the distance to it from the second place of observation (see Fig. 5-5)?

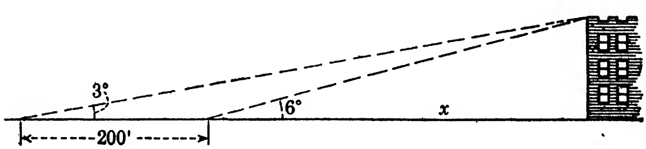


FIG. 5-5.

20. Statistics show that when a shell bursts within 50 ft. of an airplane it registers an effective hit. Find, for effective shooting, the maximum deviation from the direction that would give a central hit on an airplane distant 10,000 yd. Assume the airplane extends through a circle of diameter 75 ft.

5-3. The mil. The mil is an angular unit equal to $\frac{1}{6400}$ of four right angles.

The word mil, meaning one-thousandth, originated from the idea of adopting as a unit the angle that subtends an arc equal to $\frac{1}{1000}$ of the radius. Such an angle subtends 1 ft. at a distance of 1000 ft., 1 yd. at a distance of 1000 yd., etc. This manifestly furnishes a quick method of estimating the distance of an object whose size is known. There would under these circumstances be $2\pi/0.001$ or $6283.18+$ such units subtended by a circle. This number is too inconvenient to be of practical use in calibrating instruments. The circle is therefore divided into 6400 equal parts, and each of these is called a mil. The arc subtended by a central angle of 1 mil therefore equals $\frac{2\pi R}{6400}$ or $(0.00098+)R$, or so nearly $\frac{1}{1000}$ of the radius that it may be so taken for purposes not demanding great accuracy. This property, coupled with the knowledge that in small angles the chord very nearly equals the arc, enables us to say for rapid and rough approximation:

A mil subtends a chord equal to $\frac{1}{1000}$ of the distance to the chord.

With due regard to the degree of approximation, a small number of mils (several hundred) subtend a chord equal to the small number times $\frac{1}{1000}$ of the distance to the chord, or, in symbols

$$s = \frac{r\theta}{1000}$$

where θ is in mils and s and r are expressed in the same unit.

The methods of rapid approximate measurement of angles and distances by the use of the mil system were first developed by the Field Artillery in computing firing data. Their use was extended to mapping, sketching, and reconnaissance. During the First World War the Infantry adopted the system, and it has now become general.

The mil as a unit has the advantage that it is convenient in size for certain military measurements.

Example 1. Two points, A and B, are 50 yd. apart and 2000 yd. away. How many mils should they subtend (see Fig. 5-6)?

Solution. 50 divided by $\frac{2000}{1000}$ = **25.**

Or, at 2000 yd., 2 yd. corresponds to 1 mil; therefore 50 yd. corresponds to 25 mils.

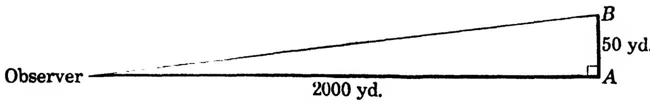


FIG. 5-6.

Example 2. An observer measures the angular distance between two points, A and B, 5000 yd. away, to be 30 mils. How far apart are A and B?

Solution. $\frac{5000}{1000} \times 30 = \mathbf{150.}$

Or, at 5000 yd., 1 mil subtends 5 yd.; therefore 30 mils subtends 150 yd.

Example 3. The angular distance between A and B is observed to be 40 mils. They are 100 yd. apart. How far away are they?

Solution. $\frac{100}{40} \times 1000 = \mathbf{2500.}$

Or 40 mils corresponds to 100 yd.; therefore 1 mil corresponds to $2\frac{1}{2}$ yd., but $2\frac{1}{2}$ is $\frac{1}{1000}$ of 2500 yd.

EXERCISES 5-3

1. A battery with a front of 60 m. is observed from a point 3000 m. away, measured on a line normal to the battery. What angle does the battery subtend? (Or what is its front in mils?)

2. A four-gun battery 4000 m. away has a front of 15 mils. How many meters between muzzles?

3. The guns in your battery have wheels $1\frac{1}{2}$ m. in diameter. You measure a wheel as 5 mils. How far are you from the battery?

4. An observer measures the front of a target to be 40 mils at a point 6000 m. away. What should a scout (*a*) 3000 m. in front of the same observer measure it to be? (*b*) 4000 m. in front of the observer?

5. A mil is $\frac{1}{1600}$ of a right angle. Find the fraction of a radian in 1 mil and the number of mils in 1 radian.

6. An enemy battery, range 6000 yd., subtends an angle of 12 mils. How many yards of front does it occupy?

7. A grade is the hundredth part of a right angle. Express an angle of 1 grade in radians. Also show that a mil is $\frac{1}{16}$ of a grade.

8. Two targets, T and t, are 20 m. apart. The range TG, perpendicular to the line of targets, is 5000 m. Two guns, G and g, are also 20 m. apart, the angle TGg being 1500 mils. Take t and g both on the same side of TG.

(*a*) What is angle tgG in order that the gun g may be laid on t?
(*b*) What change in deflection of G must be given to lay it on t?

9. A hostile trench measures 80 mils from your position. A scout 500 m. in front of you measures it 100 mils. What is the distance of the trench from your position?

10. You signal to a man at a distant tree to post himself 20 yd. from the tree (measured perpendicular to the line from the tree to you). The man is now 8 mils from the tree. How far away is the tree?

11. An observer finds that he is on the same level with the top of a distant tower that is 34 yd. high. The angular depression of the base of the tower is 8 mils. How far away is the tower?

12. From D a distant object B appears to the right of an object A, which is 6000 m. away. An observer at D measures the angle ADB to be 35 mils. He moves to C, 180 m. to the right on a line normal to AD, and measures the angle ACB to be 15 mils. How far away is B? *Hint.* The sum of the angles of a triangle is constant.

13. From Trophy Point, near the U.S. Military Academy, the angular elevation of Fort Putnam is 210 mils, and its distance is 600 yd. Also, the elevation of the top of the West Academic Building is 120 mils, and its distance is 250 yd. The West Academic Building and Fort Putnam are 500 yd. apart. What is the angular elevation of Fort Putnam as measured from the top of the West Academic Building?

14. The line of sight of a gun passes through a target 10,000 yd. away. Through an error in the sighting mechanism of the gun the plane of fire makes an angle of 10 mils with the vertical plane through the line of sight: How far from the target will the shell burst occur if the gun is correctly elevated?

5-4. The functions represented by means of lines. The circle in Fig. 5-7 is a unit circle, that is, its radius has a length of one unit. The lengths of all lines in the figure are therefore expressed in terms of the unit. CB is tangent to the circle at B, the right end of the horizontal diameter, and FG is tangent to the circle at F, the upper end of the vertical diameter. In triangle AOD, hypotenuse $OD = 1$. In triangle

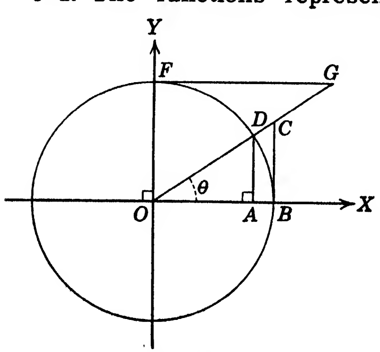


FIG. 5-7.

OBC, side $OB = 1$, and in triangle OGF, side $OF = 1$. Also, angle G equals angle θ. Hence, we obtain

$$\sin\theta = \frac{AD}{OD} = \frac{AD}{1} = AD, \qquad \cos\theta = \frac{OA}{OD} = \frac{OA}{1} = OA,$$
$$\tan\theta = \frac{BC}{OB} = \frac{BC}{1} = BC, \qquad \sec\theta = \frac{OC}{OB} = \frac{OC}{1} = OC,$$
$$\cot\theta = \frac{FG}{OF} = \frac{FG}{1} = FG, \qquad \csc\theta = \frac{OG}{OF} = \frac{OG}{1} = OG.$$

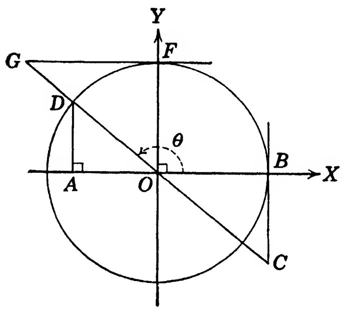


FIG. 5-8.

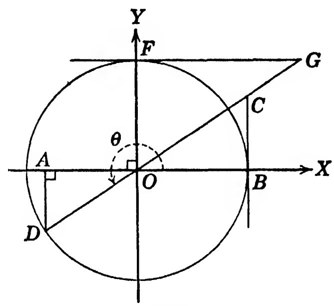


FIG. 5-9.

In Figs. 5-8, 5-9, and 5-10, θ is a second-quadrant angle, a third-quadrant angle, and a fourth-quadrant angle, respectively. The lines in these three figures are in positions analogous to the corresponding lines in Fig. 5-7. The corresponding lines are lettered in the same way. In each figure, you see the three triangles OAD, OBC, and OFG. Thus, we obtain

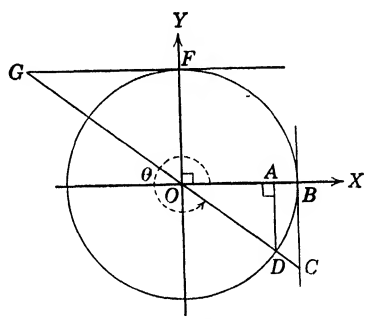


FIG. 5-10.

Quadrant II	Quadrant III	Quadrant IV
$\sin\theta = AD$	$\sin\theta = -AD$	$\sin\theta = -AD$
$\cos\theta = -OA$	$\cos\theta = -OA$	$\cos\theta = OA$
$\tan\theta = -BC$	$\tan\theta = BC$	$\tan\theta = -BC$
$\sec\theta = -OC$	$\sec\theta = -OC$	$\sec\theta = OC$
$\cot\theta = -FG$	$\cot\theta = FG$	$\cot\theta = -FG$
$\csc\theta = OG$	$\csc\theta = -OG$	$\csc\theta = -OG$

The horizontal and vertical lines are positive or negative in accordance with the system of rectangular coordinates discussed in Art. 4-2. The oblique lines are positive or negative in accordance with the following rule: **The oblique line is positive if it is extended in the direction of the terminal side of the angle θ; the oblique line is negative if it is extended in a direction opposite to that of the terminal side of the angle θ.** That is why, for example, in the third quadrant OC and OG are both negative, while in the fourth quadrant OC is positive and OG is negative.

5-5. Graph of $y = \sin x$. The graphs of the trigonometric functions are important in that they picture the variations of these functions and, at the same time, show plainly their periodic nature.

First consider the graph of $y = \sin x$. Using the table of values of trigonometric functions in Art. 2-1 and using the formulas for expressing the trigonometric functions of any angle in terms of functions of an acute angle, we make Table A.

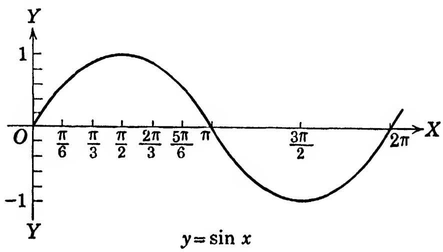
TABLE A

$x°$	x rad.	$y = \sin x$	$x°$	x rad.	$y = \sin x$
0°	0	0	210°	$7\pi/6$	−0.5
30°	$\pi/6$	0.5	240°	$4\pi/3$	−0.866
60°	$\pi/3$	0.866	270°	$3\pi/2$	−1
90°	$\pi/2$	1	300°	$5\pi/3$	−0.866
120°	$2\pi/3$	0.866	330°	$11\pi/6$	−0.5
150°	$5\pi/6$	0.5	360°	2π	0
180°	π	0			

In Fig. 5-11 you see the rectangular axes OX and OY. The plotting unit on the x-axis represents $\pi/6$ radian of angle, and five intervals represent the unit of measure to be used in laying off values of $y = \sin x$ along lines parallel to the y-axis.* This

* The unit of measure used for abscissas is not necessarily the same as the unit for ordinates.

makes it easier to measure the values of the sine in tenths. Plotting points on these axes to correspond with the pairs of values exhibited in Table A and connecting these points with a smooth curve, we obtain the graph shown in Fig. 5-11. By extending Table A indefinitely for values of x greater than 2π and for negative values of x and by plotting the corresponding points and drawing the curve through them, we should obtain

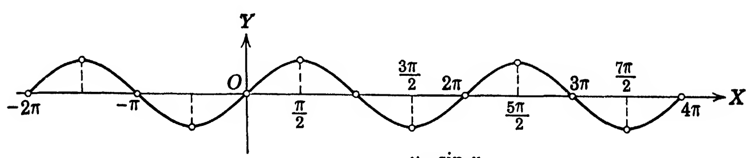


$y = \sin x$

FIG. 5-11.

both on the left and on the right of the graph drawn in Fig. 5-11 curve after curve, each having exactly the same form as the portion shown.

We know that $\sin(2\pi + x) = \sin x$; hence we conclude that when x, starting from any value, varies through 2π radians, $\sin x$ varies and takes on all of its possible values once. We express this fact by saying that $\sin x$ is periodic and has the period 2π.



$y = \sin x$

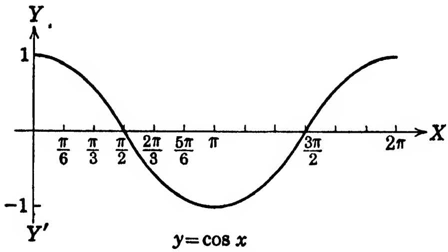
FIG. 5-12.

Figure 5-12 shows the part of the curve $y = \sin x$ corresponding to a change of three periods in x.

5-6. Graph of $y = \cos x$. Using the table of values of trigonometric functions in Art. 2-1 and using the formulas for expressing the trigonometric functions of any angle in terms of functions of an acute angle, we make Table B.

Plotting the points to correspond with the pairs of values exhibited in Table B and connecting these points with a smooth

curve, we obtain the graph shown in Fig. 5-13. The complete graph of $y = \cos x$ consists of an endless undulating curve extend-



$y = \cos x$

FIG. 5-13.

ing both to the right and to the left of the graph drawn in Fig. 5-13.*

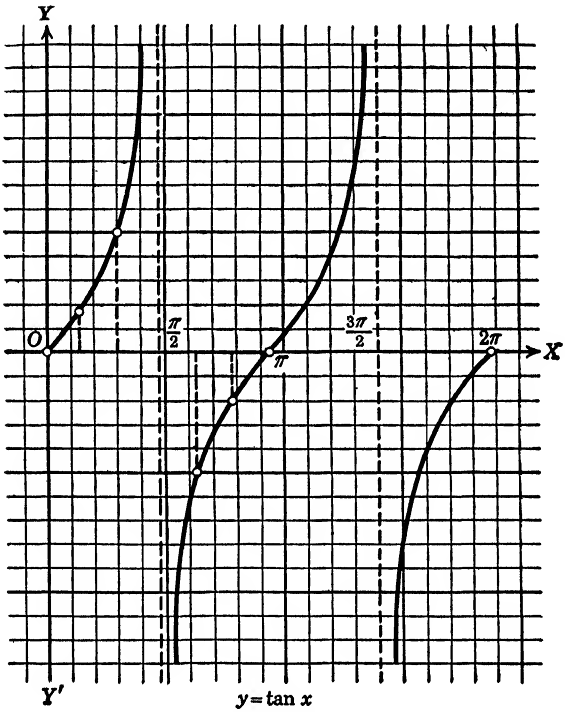
TABLE B

$x°$	x rad.	$y = \cos x$	$x°$	x rad.	$y = \cos x$
0°	0	1	210°	$7\pi/6$	−0.866
30°	$\pi/6$	0.866	240°	$4\pi/3$	−0.5
60°	$\pi/3$	0.5	270°	$3\pi/2$	0
90°	$\pi/2$	0	300°	$5\pi/3$	0.5
120°	$2\pi/3$	−0.5	330°	$11\pi/6$	0.866
150°	$5\pi/6$	−0.866	360°	2π	1
180°	π	−1			

Since $\cos (2\pi + x) = \cos x$, we conclude that $\cos x$ is periodic and has the period 2π.

5-7. Graph of $y = \tan x$. The Table C of values applies to $y = \tan x$, and Fig. 5-14 shows the corresponding graph. The

* Since $\cos x = \sin \left(\frac{\pi}{2} - x\right)$, it appears that the cosine curve has the same form as the sine curve. In fact, if the cosine curve is translated as a whole $\pi/2$ units parallel to the x-axis, it will coincide with the sine curve.



$y = \tan x$
Fig. 5-14.

Table C

$x°$	x rad.	$y = \tan x$
0°	0	0
30°	$\pi/6$	0.577
60°	$\pi/3$	1.732
90°	$\pi/2$	∞
120°	$2\pi/3$	-1.732
150°	$5\pi/6$	-0.577
180°	π	0

$x°$	x rad.	$y = \tan x$
210°	$7\pi/6$	0.577
240°	$4\pi/3$	1.732
270°	$3\pi/2$	∞
300°	$5\pi/3$	-1.732
330°	$11\pi/6$	-0.577
360°	2π	0

straight line perpendicular to the x-axis at $x = \pi/2$ is drawn to indicate that, as the abscissa of a moving point on the curve approaches $\pi/2$ as a limit, the point on the curve approaches indefinitely close to the line, and the length of the ordinate of the point becomes greater and greater without limit. The other line perpendicular to the x-axis where $x = 3\pi/2$ indicates the same kind of situation. Both the table of values and the graph show that the part of the curve from π to 2π has the same form as the part from 0 to π. This follows also from the fact that $\tan x = \tan (\pi + x)$. The complete curve consists of an endless number of branches having the same form as the branch corresponding to the values of x from $\pi/2$ to $3\pi/2$. From this discussion it appears that $\tan x$ is periodic and has the period π.

5-8. Graphs of $y = \cot x$, $y = \sec x$, $y = \csc x$. The graphs of $y = \cot x$ (see Fig. 5-15), $y = \sec x$ (see Fig. 5-16), $y = \csc x$

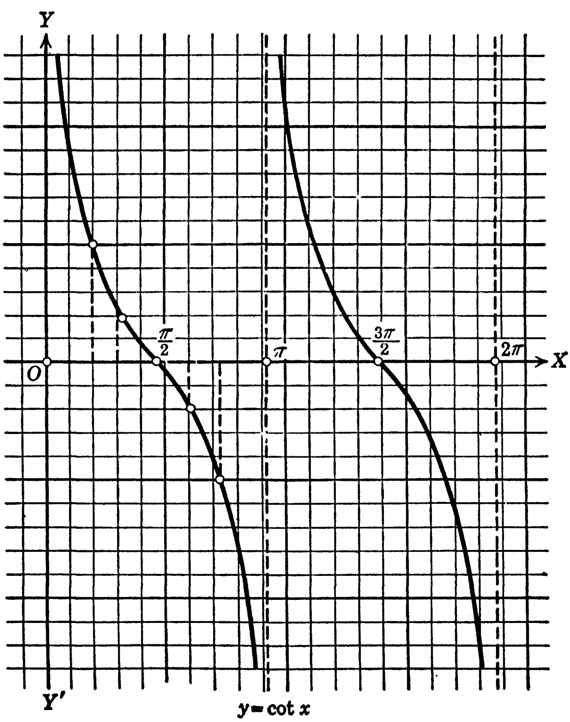
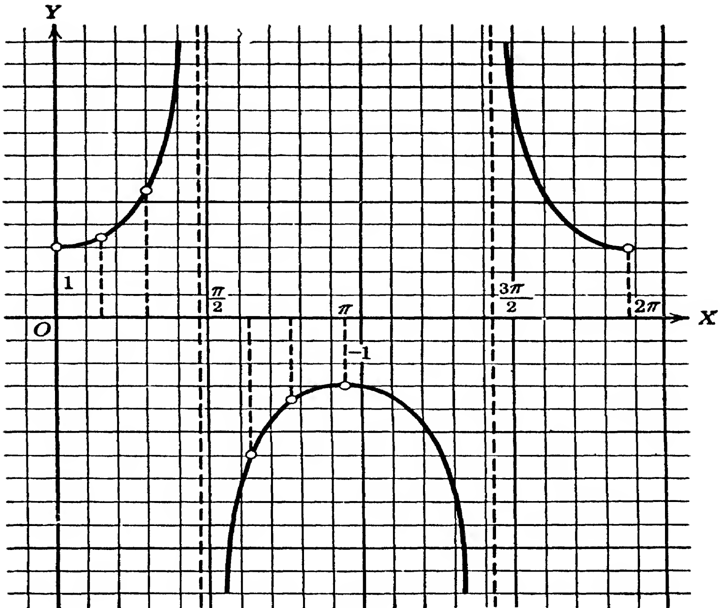
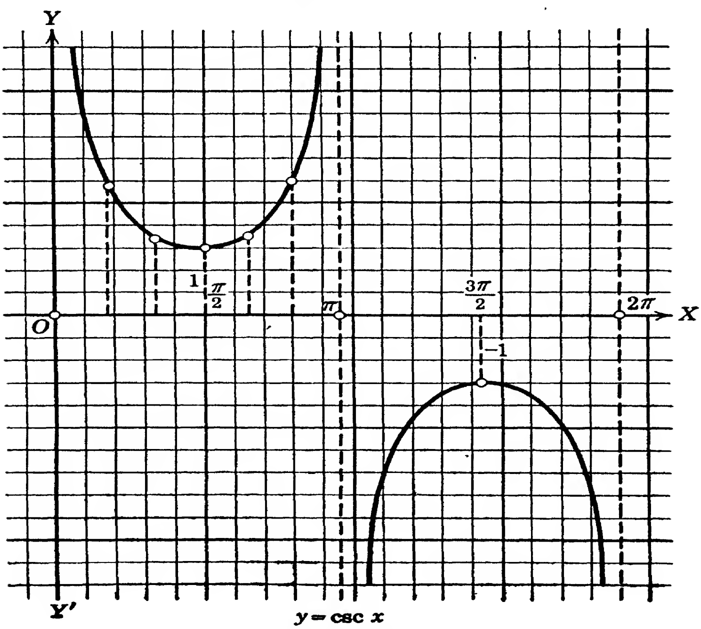


FIG. 5-15.



$y = \sec x$

FIG. 5-16.



$y = \csc x$

FIG. 5-17.

(see Fig. 5-17) are obtained from the sets of values shown in the following table:

Table D

$x°$	x rad.	$y = \cot x$	$y = \sec x$	$y = \csc x$
0°	0	∞	1	∞
30°	$\pi/6$	1.732	1.155	2
60°	$\pi/3$	0.577	2	1.155
90°	$\pi/2$	0	∞	1
120°	$2\pi/3$	-0.577	-2	1.155
150°	$5\pi/6$	-1.732	-1.155	2
180°	π	∞	-1	∞
210°	$7\pi/6$	1.732	-1.155	-2
240°	$4\pi/3$	0.577	-2	-1.155
270°	$3\pi/2$	0	$-\infty$	-1
300°	$5\pi/3$	-0.577	2	-1.155
330°	$11\pi/6$	-1.732	1.155	-2
360°	2π	∞	1	∞

In every case the complete graph consists of an endless number of parts, each congruent with the part shown.

It is easily seen that each of the functions graphed has the same period as its reciprocal function.

5-9. Graphs and periods of the trigonometric functions of $k\theta$. First consider the graph of $y = \sin 2x$. The values in Table E are found as in the preceding articles.

Plotting the corresponding points and connecting them with a smooth curve, we have Fig. 5-18.

From Table E as well as from Fig. 5-18 it appears that

$$y = \sin 2x$$

TABLE E

x rad.	$x°$	$2x°$	$y = \sin 2x$
0	0°	0°	0
$\pi/6$	30°	60°	0.866
$\pi/3$	60°	120°	0.866
$\pi/2$	90°	180°	0
$2\pi/3$	120°	240°	−0.866
$5\pi/6$	150°	300°	−0.866
π	180°	360°	0
$7\pi/6$	210°	420°	0.866
$4\pi/3$	240°	480°	0.866
$3\pi/2$	270°	540°	0
$5\pi/3$	300°	600°	−0.866
$11\pi/6$	330°	660°	−0.866
2π	360°	720°	0

has taken its complete set of values twice, once while x passed from 0 to π and once while x passed from π to 2π. Hence we

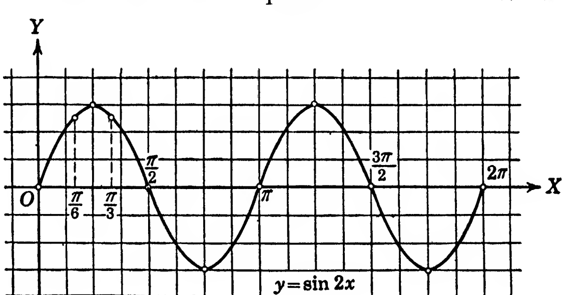


FIG. 5-18.

conclude that the period of $\sin 2x$ is $2\pi/2 = \pi$. Since $2x$ passed through 2π radians while x passed through π radians, the period

of sin $2x$ is one-half the period of sin x. Similarly it appears that kx would pass through 2π radians while x passed through $2\pi/k$ radians; *hence the period of sin kx is $2\pi/k$.* A like argument would show that *the period of cos kx is $2\pi/k$, the period of tan kx is π/k,* and each reciprocal function has the same period as the function of which it is the reciprocal.

In plotting $y = \sin kx$ and $y = \cos kx$, we observe that the greatest value that y may have is unity. Evidently, if we should plot $y = a \sin kx$ or $y = a \cos kx$, the greatest value y could attain in either case would be a. This number a is spoken of as the *amplitude* of y.

EXERCISES 5-4

1. Find the period of each of the following functions.

(*a*) $\sin 5\theta$.
(*b*) $3 \cos 8\theta$.
(*c*) $2 \tan \frac{1}{2}\theta$.
(*d*) $\frac{1}{2} \cot 4\theta$.
(*e*) $2 \sec 6\theta$.
(*f*) $242 \csc 2\theta$.
(*g*) $5 \cos (4\theta + 60°)$.
(*h*) $5 \tan \pi\theta$.
(*i*) $3 \cot \frac{1}{3}\varphi$.
(*j*) $7.9 \sec (3\varphi - 45°)$.
(*k*) $2 + \sin 3\varphi$.
(*l*) $6 + \cos 2\varphi$.
(*m*) $-6 \tan \varphi$.
(*n*) $112 \sin (277\theta + 30°)$.

2. Find the amplitude of each of the following functions:

(*a*) $\sin 6\varphi$.
(*b*) $4 \cos 6\varphi$.
(*c*) $\frac{1}{2} \sin \frac{1}{2}\varphi$.
(*d*) $8.6 \cos \varphi$.
(*e*) $334 \cos (\varphi + 60°)$.
(*f*) $\frac{3}{16} \cos (\varphi - \pi)$.
(*g*) $\cos (2 + \theta)$
(*h*) $8 \sin (241\theta - 45°)$.

3. Plot:

(*a*) $y = \cos x$.
(*b*) $y = 2 \sin x$.
(*c*) $y = 2 \tan x$.
(*d*) $y = 3 \cot x$.
(*e*) $y = 4 \csc x$.
(*f*) $y = 5 \sec x$.
(*g*) $y = 2 \sin 2x$.
(*h*) $y = 4 \tan 2x$.
(*i*) $2y = \cos 2x$.
(*j*) $y = \tan \frac{1}{2}x$.
(*k*) $y = \sin \frac{2x}{3}$.
(*l*) $y = \cos \frac{x}{4}$.
(*m*) $2y = \cot \frac{x}{4}$.
(*n*) $y = \sec (x + \pi)$.
(*o*) $y = \csc \left(\frac{\pi}{2} + \theta\right)$.

4. Plot on the same set of axes:

(*a*) $y = \cos x$ and $y = \cos 2x$.
(*b*) $y = \sin x$ and $y = 2 \sin x$.

(*c*) $y = \tan x$ and $y = \cot x$.
(*d*) $y = 2 \sin x$ and $y = 2 \csc x$.
(*e*) $y = \sin 2x$ and $y = \cos \frac{1}{2}x$.
(*f*) $y = 2 \tan 2x$ and $y = \cot \frac{1}{2}x$.

5. Plot the graph of each of the following equations for the indicated range of values of x:

(*a*) $y = \sin x + \cos x$, 0 to 2π.
(*b*) $y = 3 \cos x + 2 \sin x$, $-\pi$ to 2π.
(*c*) $y = \cos x + 3 \sin 2x$, $-\pi$ to π.
(*d*) $y = \sin x - \cos x$, $-\pi$ to π.
(*e*) $y = \sin \frac{1}{2}x - 2 \cos x$, -2π to 2π.

6. By plotting the graph of $y = \sin x$ and using $\csc x = 1/\sin x$, obtain the graph of $y = \csc x$ on the same set of axes and to the same scale.

7. By plotting the graph of $y = \cos x$ and using $\sec x = 1/\cos x$, obtain the graph of $y = \sec x$ on the same set of axes and to the same scale.

8. Plot the curve $y = \sin 3x$. Then construct the curve $y = \csc 3x$ on the same graph by taking account of the fact that $\csc 3x$ and $\sin 3x$ are reciprocal functions.

9. Plot one period of the graph of each of the following equations on the same set of axes and to the same scale:

(*a*) $y = \sin x$, $y = \sin 2x$, and $y = \sin \frac{1}{2}x$.
(*b*) $y = \sin x$, $y = 2 \sin x$, and $y = \frac{1}{2} \sin x$.
(*c*) $y = \cos x$, $y = \cos 2x$, and $y = 2 \cos x$.
(*d*) $y = \cos x$, $y = \frac{1}{2} \cos x$, and $y = \cos \frac{3}{2}x$.

10. If t stands for time in seconds and y for magnitude in volts, then the equation

$$y = 110 \sin 377t$$

represents the voltage causing an alternating current of electricity. Find the period and the maximum magnitude of the voltage.

MISCELLANEOUS EXERCISES 5-5

1. Express the following angles in radians: 10°, 30°, 45°, 135°, 225°, −270°, −18°, −24°15′.

2. Construct approximately the following angles: 2 radians, $3\frac{1}{2}$ radians, $-\frac{1}{2}$ radian, −4 radians, 9 radians.

3. Construct the following angles:

$$\frac{\pi}{2}, \ -\frac{\pi}{3}, \frac{\pi}{4}, \pi, \ -\frac{5\pi}{4}, \frac{5\pi}{2}.$$

4. Express the following angles in degrees: $\frac{\pi}{3}$ radians, π radians, $\frac{2}{3}\pi$ radians, $\frac{7}{4}\pi$ radians, 2 radians, 5 radians, -3 radians.

5. Express the following as functions of an acute angle less than 45°:

(*a*) $\cot \frac{8\pi}{3}$. (*b*) $\sin \frac{37\pi}{14}$.

(*c*) $\tan \frac{17\pi}{10}$. (*d*) $\sec \frac{9\pi}{14}$.

6. In a circle whose radius is 5, the length of an intercepted arc is 12. Find the corresponding central angle (*a*) in radians; (*b*) in degrees.

7. In a circle of radius 12 ft., find the length of the arc intercepted by a central angle of 16°.

8. Find the angle between the tangents to a circle at two points whose distance apart measured on the arc of the circle is 378 ft., the radius of the circle being 900 ft.

9. Assuming the earth's orbit to be a circle of radius 92,000,000 miles, what is the velocity of the earth in its path in miles per second?

10. A belt travels around two pulleys whose diameters are 3 ft. and 10 in., respectively. The larger pulley makes 80 revolutions per minute. Find the angular velocity of the smaller pulley in radians per second; also the speed of the belt in feet per minute.

11. Find the numerical value of

(*a*) $\cos 30° + \cos 150° + \tan 60° + \tan 120°$.
(*b*) $(\tan 120° - \tan 135°) \times (\tan 120° + \tan 135°)$.
(*c*) $\sin 420° \cdot \cos 390° + \cos (-300°) \cdot \sin (-330°)$.
(*d*) $\cos 570° \cdot \sin 510° - \sin 330° \cdot \cos 390°$.
(*e*) $\tan \frac{2\pi}{3} - \sin \frac{7\pi}{6} + \sec \frac{3\pi}{4} - \csc^2 \frac{5\pi}{3}$.
(*f*) $3 \tan 210° + 2 \tan 120°$.
(*g*) $5 \sec^2 135° - 6 \cot^2 300°$.

12. Simplify each of the following expressions:

(*a*) $\cos \left(\frac{\pi}{2} + x\right) \sin (3\pi - x) - \cos (2\pi + x) \sin \left(\frac{3\pi}{2} - x\right)$.

(*b*) $\sec (180° - \theta) \times \cos \theta \times \tan (180° - \theta) \times \cot \theta$.

(*c*) $\dfrac{\cos(90^\circ - A)}{\sin(180^\circ + A)} + \dfrac{\cos A}{\sin(90^\circ + A)} + \dfrac{\tan(270^\circ + A)}{\tan(-A)}$.

(*d*) $\sec(180^\circ + \theta)\csc(270^\circ + \theta) + \tan(180^\circ - \theta)\cot(270^\circ - \theta)$.

(*e*) $\dfrac{\cos(180^\circ - \theta)}{\sin(90^\circ - \theta)} + \dfrac{\cot(270^\circ + \theta)\cos(270^\circ - \theta)}{\sec(-\theta)}$.

(*f*) $\dfrac{\cos(90^\circ + \alpha)}{\sin(-\alpha)} + \dfrac{\tan(-\alpha)}{\tan(180^\circ + \alpha)}$.

(*g*) $\dfrac{\sin(180^\circ - \theta)}{\cos(90^\circ + \theta)} \times \dfrac{\tan(180^\circ + \theta)}{\cot(90^\circ + \theta)}$.

13. Prove:

(*a*) $\cos(90^\circ + \theta)/\tan(180^\circ + \theta) = 1/\csc(270^\circ - \theta)$.

(*b*) $\dfrac{\tan(180^\circ + \alpha) - \tan(180^\circ - \beta)}{\tan(270^\circ - \alpha) - \cot(-\beta)} = \tan\alpha\tan\beta$.

(*c*) $\dfrac{\tan 3\pi - \tan 2\theta}{1 + \tan 3\pi\tan 2\theta} = \tan(3\pi - 2\theta)$.

(*d*) $(a - b)\tan(90^\circ - x) + (a + b)\cot(90^\circ + x) = (a - b)\cot x - (a + b)\tan x$.

(*e*) $\sin\left(\frac{\pi}{2} + x\right)\sin(\pi + x) + \cos\left(\frac{\pi}{2} + x\right)\cos(\pi - x) = 0$.

(*f*) $\cos(\pi + x)\cos\left(\frac{3\pi}{2} - y\right) - \sin(\pi + x)\sin\left(\frac{3\pi}{2} - y\right) = \cos x\sin y - \sin x\cos y$.

(*g*) $\tan x + \tan(-y) - \tan(\pi - y) = \tan x$.

14. If $\cot 260^\circ = +a$, prove that $\cos 350^\circ = +\dfrac{1}{\sqrt{1 + a^2}}$.

15. If $\sec 340^\circ = +a$, prove that $\sin 110^\circ = \dfrac{1}{a}$, and than $110^\circ = -\dfrac{1}{\sqrt{a^2 - 1}}$.

16. If $\cos 300^\circ = +a$, prove that $\cot 120^\circ = -\dfrac{a}{\sqrt{1 - a^2}}$.

17. Show that $\cot(270^\circ + x)$ is equal to the negative of the cotangent of the supplementary angle.

18. If $\tan 310^\circ = c$, find $\dfrac{\sin 320^\circ - \cos 310^\circ}{\tan 140^\circ + \cot 220^\circ}$ in terms of c.

19. If $\sin\theta = -\frac{15}{17}$ and θ is in the third quadrant, find the functions of $(-\theta)$.

20. If $\cot(-\theta) = 2$ and θ is in the second quadrant, find the functions of θ.

21. If $\cos \alpha = -\frac{5}{13}$ and α is in the second quadrant, evaluate:

$$\frac{\sin (180° - \alpha)}{\sec (270° + \alpha)} + \frac{\cos (360° - \alpha)}{\csc (270° - \alpha)}.$$

22. Tan $\beta = \frac{3}{4}$ and β is in the third quadrant, evaluate:

$$\frac{\sin (-\beta) \csc^2 (180° + \beta)}{\sec^2 (90° + \beta)} - \frac{\cot (270° + \beta)}{\tan (180° - \beta)}.$$

23. Plot $y = \sin 2x$.

24. Plot $y = 3 \cos x$.

25. Plot $y = \tan \frac{1}{2}x$.

26. Plot $y = \cos 2x$ and $y = \sec 2x$ on the same set of axes.

27. Express in radians the sum of the angles of a convex polygon of n sides.

28. The rotor of a steam turbine is 2 ft. in diameter and makes 2500 revolutions per minute. The blades of the turbine, situated on the circumference of the rotor, have one-half the velocity of the steam that drives them. What is the velocity of the steam in feet per second?

29. The diameter of the sun is approximately 864,000 miles and at a certain instant it subtends an angle of $32'$ at a point on the earth. Compute the approximate distance from the earth to the sun at this instant.

30. Assuming that the diameter of the smallest sphere clearly visible to the ordinary eye subtends an angle of $1'$ at the eye, find the greatest distance at which a baseball 2.9 in. in diameter can be clearly seen.

31. A horse is tethered to a stake at the corner of a field where the boundaries intersect at an angle of 75°. How long should the rope be so that the horse can graze over half an acre?

32. Find the length in feet of an arc of $3''$ on the earth's equator.

CHAPTER 6

GENERAL FORMULAS

6-1. The addition formulas. In many respects, the two formulas,

$$\left.\begin{aligned} \sin{(A + B)} &= \sin A \cos B + \cos A \sin B, \\ \cos{(A + B)} &= \cos A \cos B - \sin B \sin B, \end{aligned}\right\} \qquad (1)$$

are the most important ones in trigonometry. They are called the addition formulas because they express trigonometric functions of the sum of two angles in terms of the trigonometric functions of the angles. These formulas, holding true as they do for all angles, positive and negative, are the basis of trigonometric analysis. It will appear in what follows that all the formulas of this chapter and many others are derived from them.

6-2. Proof of the addition formulas. Special case. We shall first prove formulas (1) for the case when both angles A and B are positive acute angles and $A + B < 90°$. In Fig. 6-1 angles A and B appear as adjacent angles with common vertex O and common side OC. Point D is taken on the terminal side of angle B so that OD is 1 unit long, DC is drawn perpendicular to OC, DG and CE perpendicular to OX, and FC perpendicular to GD.

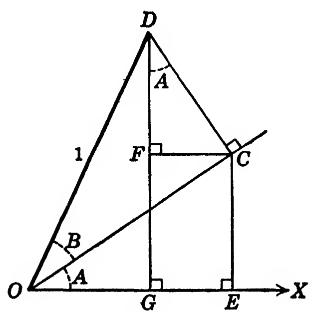


Fig. 6-1.

The proof of formulas (1) will consist in finding the lengths of the line segments in Fig. 6-1, writing them on the figure to obtain Fig. 6-2, and then reading the formulas from Fig. 6-2. The student may do this for himself without reading the following development:

From Fig. 6-1 we read

$$\frac{CD}{1} = \sin B, \qquad \frac{OC}{1} = \cos B. \qquad (2)$$

Angle FDC is equal to angle A because its sides are respectively perpendicular to the sides of angle A. Hence, from triangle FCD,

$$\frac{FC}{CD} = \sin A, \qquad \frac{FD}{CD} = \cos A. \tag{3}$$

Replacing CD in (3) by its value $\sin B$ from (2) and multiplying both members of each equation by $\sin B$, we obtain

$$FC = \sin A \sin B, \qquad FD = \cos A \sin B. \tag{4}$$

From triangle OEC,

$$\frac{EC}{OC} = \sin A, \qquad \frac{OE}{OC} = \cos A. \tag{5}$$

Replacing OC in (5) by its value $\cos B$ from (2) and multiplying

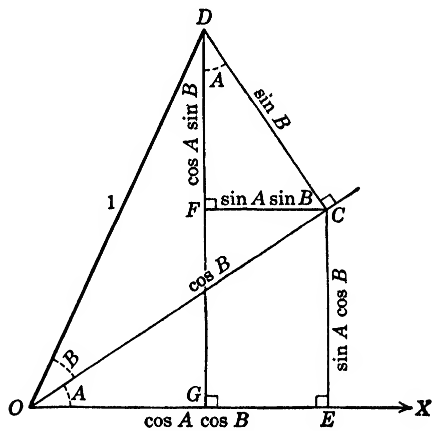


FIG. 6-2.

both members of each equation by $\cos B$, we get

$$EC = \sin A \cos B, \qquad EO = \cos A \cos B. \tag{6}$$

Figure 6-2 is the result of writing on each line in Fig. 6-1 its value obtained from one of the equations (2), (4), (5), and (6). Noting that

$$\sin (A + B) = \frac{GD}{1} = EC + FD$$

and

$$\cos (A + B) = \frac{OG}{1} = OE - FC,$$

we read from Fig. 6-2

$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B, \qquad (7)\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B. \qquad (8)\end{aligned}$$

That the formulas (7) and (8) are true for all values of A and B will be proved in the next article. We shall now assume that they are generally true and use them to obtain two other closely related formulas. Replacing B by $-B$ in (7) and (8), we get

$$\left.\begin{aligned} \sin[A+(-B)] &= \sin A\cos(-B) + \cos A\sin(-B),\\ \cos[A+(-B)] &= \cos A\cos(-B) - \sin A\sin(-B).\end{aligned}\right\} \qquad (9)$$

In accordance with Art. 4-9,

$$\cos(-B) = \cos B \qquad \text{and} \qquad \sin(-B) = -\sin B.$$

Replacing $\cos(-B)$ by $\cos B$ and $\sin(-B)$ by $-\sin B$ in (9), we obtain

$$\begin{aligned} \sin(A-B) &= \sin A\cos B - \cos A\sin B, \qquad (10)\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B. \qquad (11)\end{aligned}$$

Example. Use (8) to find $\cos 75°$.

Solution. Substituting $45°$ for A and $30°$ for B in (8), we obtain

$$\cos 75° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

EXERCISES 6-1

1. Use (1) to find $\sin(A+B)$ and $\cos(A+B)$ if $\sin A = \frac{1}{3}$ and $\cos B = \frac{2}{3}$, and if A and B are both acute angles.

2. Substitute $A = 30°$, $B = 60°$ in (1) to obtain $\sin 90°$ and $\cos 90°$.

3. Substitute $A = 30°$, $B = 45°$ in (1) to obtain $\sin 75°$ and $\cos 75°$. Then write the values of the trigonometric functions of $75°$.

4. By using (1), find $\sin 105°$ and then find the values of the other trigonometric functions of $105°$ from a right triangle.

5. Given that α and β terminate in the second and in the fourth quadrant, respectively, and that $\sin\alpha = \cos\beta = \frac{3}{5}$, find $\cos(\alpha+\beta)$.

6. Using the table of natural functions, find (*a*) $\sin 31°$ from the functions of $20°$ and $11°$; (*b*) the difference between $\sin(20° + 11°)$ and $\sin 20° + \sin 11°$.

7. Find $\cos(A + B)$ if $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, A and B being positive acute angles.

8. If $\tan x = \frac{3}{4}$ and $\tan y = \frac{7}{24}$, find $\sin(x + y)$ and $\cos(x + y)$ when x and y are acute angles.

9. Set $B = A$ in (1) to obtain $\sin 2A$ and $\cos 2A$ in terms of $\sin A$ and $\cos A$.

10. Set $A = 90°$ in (1) and check the result by the methods of Art. 4-11.

11. Find, by using formulas (7) to (11), the sine and cosine of

(*a*) $90° + y$.	(*b*) $180° - y$.	(*c*) $180° + y$.
(*d*) $270° - y$.	(*e*) $270° + y$.	(*f*) $360° - y$.
(*g*) $360° + y$.	(*h*) $x - 90°$.	(*i*) $x - 180°$.
(*j*) $x - 270°$.	(*k*) $-y$.	(*l*) $45° - y$.
(*m*) $45° + y$.	(*n*) $30° + y$.	(*o*) $60° - y$.

12. Show that

$$\sin(45° - x) = \frac{\cos x - \sin x}{\sqrt{2}}.$$

13. Show that

$$\cos(210° + x) = \tfrac{1}{2}(\sin x - \sqrt{3}\cos x).$$

14. Show that

$$\cos(60° + \alpha) = \frac{\cos\alpha - \sqrt{3}\sin\alpha}{2}.$$

15. Find $\cos(210° + A)$ if $\sec A = -\sqrt{3}$ and A is a second-quadrant angle.

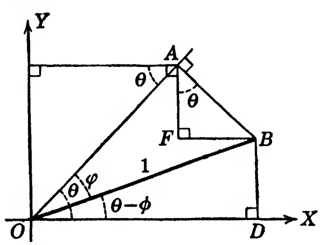


Fig. 6-3.

16. In Fig. 6-3 let $OB = 1$ unit and express all its line segments in terms of trigonometric functions of θ and φ. Then deduce the formulas

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi,$$
$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi.$$

17. Show that

$$\sin(0 - 120°) = -\frac{\sin\beta + \sqrt{3}\cos\beta}{2}.$$

18. Show that

$$\sin(45° + x) = \frac{\cos x + \sin x}{\sqrt{2}}.$$

19. Show that

$$\sin (y + 135°) = \frac{\cos y - \sin y}{\sqrt{2}}.$$

20. Show that

$$\cos (A - B) \cos (A + B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

21. Show that

$$\sin (x + y) \cos y - \cos (x + y) \sin y = \sin x.$$

22. Show that

$$\sin (x + 60°) - \cos (x + 30°) = \sin x.$$

23. Use (1) to prove that

(*a*) $\sin 2x = 2 \sin x \cos x.$
(*b*) $\cos 2x = \cos^2 x - \sin^2 x.$
(*c*) $\sin 3x = \sin x \cos 2x + \cos x \sin 2x.$
(*d*) $\sin 3x = \sin 5x \cos 2x - \cos 5x \sin 2x.$

24. Express $\sin 3\theta$ in terms of $\sin \theta$.

25. Express $\cos 3\theta$ in terms of $\cos \theta$.

26. Prove that

$$\frac{\sin (\alpha + \beta) + \sin (\alpha - \beta)}{\cos (\alpha + \beta) + \cos (\alpha - \beta)} = \tan \alpha.$$

6-3. Removal of restrictions on the addition formulas. In Art. 6-2 the angles A and B were assumed to be acute angles such that $A + B$ was less than 90°. This article is designed to show that formulas (1) hold true when angles A and B are unrestricted in magnitude and sign.

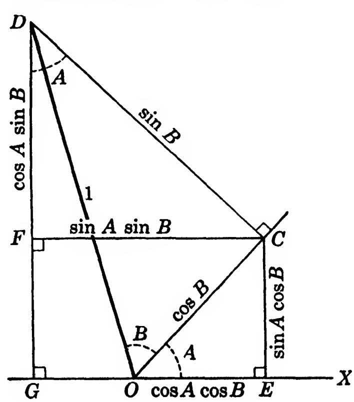


FIG. 6-4.

The proof given in Art. 6-2 applies equally well to Fig. 6-4. Hence formulas (1) are true when A and B are any two acute angles.

Let A be an angle greater than 90° but less than 180°, and let B be a positive acute angle. Let

$$A' = A - 90°. \tag{12}$$

Since A' and B are acute angles, formulas (1) hold true for them, and

$$\left.\begin{aligned}\sin (A' + B) &= \sin A' \cos B + \cos A' \sin B,\\ \cos (A' + B) &= \cos A' \cos B - \sin A' \sin B.\end{aligned}\right\} \tag{13}$$

Replacing A' in (13) by $A - 90°$ from (12) and using the methods of Chap. 5, we have

$$\left.\begin{aligned}\sin (A' + B) &= \sin (A + B - 90°) = -\cos (A + B),\\ \cos (A' + B) &= \cos (A + B - 90°) = \sin (A + B),\\ \sin A' &= \sin (A - 90°) = -\cos A,\\ \cos A' &= \cos (A - 90°) = \sin A.\end{aligned}\right\} \tag{14}$$

Substituting the values of $\sin (A' + B)$, $\cos (A' + B)$, $\sin A'$, and $\cos A'$ from (14) in (13), we obtain, after slight simplification,

$$\begin{aligned}\cos (A + B) &= \cos A \cos B - \sin A \sin B,\\ \sin (A + B) &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

Hence it appears that formulas (1) hold true when A is an obtuse angle and B an acute angle.

We next let A be an angle greater than 180° but less than 270° and let B be an acute angle. By letting $A' = A - 90°$ and arguing as above, we prove that formulas (1) hold true for this new case. By continuing this process indefinitely we can show that (1) holds true when A is any positive angle and B is a positive acute angle. Again, letting A be any angle and B an angle greater than 90° but less than 180°, we argue as above and show that (1) holds true in this case. Continuing this process with reference to B, we finally deduce that (1) holds true when A and B are any positive angles.

If (1) holds true for any pair of positive angles A and B, evidently it will still hold true if A and B be decreased by any multiples of 360°. Since any negative angle may be obtained by subtracting some multiple of 360° from a suitable positive angle, and since (1) holds true when A and B are any positive angles, it appears that (1) holds true when A and B represent any negative angles. Hence (1) holds true when A and B represent any angles.

6-4. Addition and subtraction formulas for the tangent. By using (1), we may deduce addition formulas for the other functions. To express $\tan (A + B)$ in terms of $\tan A$ and $\tan B$ we

have

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}. \tag{15}$$

Dividing numerator and denominator of the right-hand member of (15) by $\cos A \cos B$, we obtain

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}},$$

or

$$\mathbf{\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.} \tag{16}$$

Since equations (1) hold true for all values of A and B, it follows that (16) holds true for all values of A and B for which $\tan(A + B)$ is defined. Replacing B by $-B$ and therefore $\tan B$ by $\tan(-B) = -\tan B$ in (16), we obtain

$$\mathbf{\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.} \tag{17}$$

Addition and subtraction formulas for the other functions could be obtained by a similar procedure.

EXERCISES 6-2

1. Express the tangent functions in (16) in terms of cotangent functions, and thus deduce that

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}.$$

2. Prove the formula of Exercise 1 by starting from formulas (1).

3. Find $\tan 105°$ in the form of radicals by using (16).

4. Check (16) by substituting in it $A = 4\pi/3$, $B = 3\pi/4$.

5. If $\tan \alpha = \frac{3}{4}$ and $\sin \beta = \frac{12}{13}$, find the functions of $\alpha + \beta$ when α is of the third and β of the second quadrant.

6. If $\cos \alpha = -\frac{40}{41}$ and $\sin \beta = -\frac{5}{13}$, find the functions of $\alpha - \beta$ when α is of the third, and β of the fourth quadrant.

7. If $\tan x = \frac{1}{3}$ and $x - y = 45°$, find $\tan y$.

8. If $\tan y = 2$ and $x + y = 135°$, find $\tan x$.

9. Show that

$$\tan(A - 60°) = \frac{\tan A - \sqrt{3}}{1 + \sqrt{3}\tan A}.$$

10. Show that

$$\tan (x + 45°) + \cot (x - 45°) = 0.$$

11. Show that

$$\cot A - \cot B = \frac{\sin (B - A)}{\sin A \sin B}.$$

12. Show that

$$\frac{\cot (45° - y)}{\cot (45° + y)} = \frac{1 + 2 \sin y \cos y}{1 - 2 \sin y \cos y}.$$

13. In Fig. 6-1 let $OE = 1$ unit, and express all its line segments in terms of trigonometric functions of A and B. Then deduce formulas (16) and (17).

14. Use (1), (10), and (11) to simplify

(*a*) $\sin 3x \cos 2x + \cos 3x \sin 2x$.
(*b*) $\cos 3x \cos 2x + \sin 3x \sin 2x$.
(*c*) $\sin 3x \cos 2x - \cos 3x \sin 2x$.
(*d*) $\cos (x + 45°) \cos (45° - x) - \sin (x + 45°) \sin (45° - x)$.
(*e*) $\cos^2 x - \sin^2 x$.
(*f*) $\sin x \cos x + \cos x \sin x$.

15. Use (16) to simplify

(*a*) $\dfrac{\tan 3x + \tan 2x}{1 - \tan 2x \tan 3x}$. (*b*) $\dfrac{2 \tan x}{1 - \tan^2 x}$.

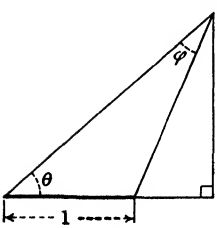


FIG. 6-5.

16. Express all line segments of Fig. 6-5 in terms of θ and φ, and from the results deduce a formula for $\sin (\theta + \varphi)$ and a formula for $\cos (\theta + \varphi)$.

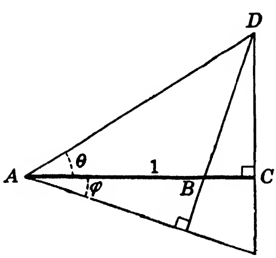


FIG. 6-6.

17. Taking AC of Fig. 6-6 equal to 1 unit, express all line segments of the figure in terms of θ and φ, and from your results deduce formula (16).

Hint. Angle $BDC = \varphi$.

18. Taking BC of Fig. 6-6 equal to 1 unit, deduce from the figure the formula of Exercise 1.

19. Prove the following identities:

(*a*) $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

(*b*) $\tan(45° - x)\tan(135° - x) = -1$.

(*c*) $\cos(60° + x)\cos(30° + x) + \sin(60° + x)\sin(30° + x) = \dfrac{\sqrt{3}}{2}$.

(*d*) $\cos 5x \cos 3x + \sin 5x \sin 3x = 2\cos^2 x - 1$.

(*e*) $\dfrac{\sin(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{\cot\alpha + \cot\beta}{1 + \cot\alpha\cot\beta}$.

(*f*) $\csc 2\theta = \cot\theta - \cot 2\theta$.

20. The expression $a\sin\theta + b\cos\theta$ may be written in the form

$$\sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}}\sin\theta + \frac{b}{\sqrt{a^2 + b^2}}\cos\theta\right).$$

Hence if we let $\tan\alpha = b/a$, we have

$$a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}\,(\sin\theta\cos\alpha + \cos\theta\sin\alpha),$$

or

$$a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}\sin(\theta + \alpha). \qquad (A)$$

Write each of the following expressions in the form (*A*):

(*a*) $2\sqrt{3}\sin\theta + 2\cos\theta$.

(*b*) $a\sin\theta + a\cos\theta$.

(*c*) $\dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{\sqrt{2}}\cos\theta$.

(*d*) $3\sin\theta - \sqrt{3}\cos\theta$.

(*e*) $3\sin\theta + 4\cos\theta$.

(*f*) $\sqrt{2}\cos\theta - \sqrt{2}\sin\theta$.

21. Show that

$$\sin(A + B + C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C.$$

Hint. $A + B + C = (A + B) + C$.

22. Show that

$$\cos(A + B + C) = \cos A\cos B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C - \sin A\sin B\cos C.$$

6-5. The double-angle formulas and the half-angle formulas. To express the trigonometric functions of 2θ in terms of functions of θ replace φ by θ in the addition formulas. Thus, to find $\sin 2\theta$,

substitute θ for ϕ in the formula

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and obtain

$$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

or

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{18}$$

Similarly, from the formula

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

we obtain

$$\cos 2\theta = \cos^2\theta - \sin^2\theta. \tag{19}$$

By using the fact that $\sin^2\theta + \cos^2\theta = 1$, we easily deduce from (19)

$$\cos 2\theta = 2\cos^2\theta - 1, \tag{20}$$
$$\cos 2\theta = 1 - 2\sin^2\theta. \tag{21}$$

From formula (16), we obtain

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}. \tag{22}$$

Solving (20) for $\cos\theta$ and (21) for $\sin\theta$, we obtain

$$\cos\theta = \pm\sqrt{\frac{1 + \cos 2\theta}{2}}, \qquad \sin\theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}. \tag{23}$$

To get half-angle formulas, replace θ by $\frac{1}{2}\varphi$ in (23) and obtain

$$\left.\begin{aligned} \sin\tfrac{1}{2}\varphi &= \pm\sqrt{\frac{1 - \cos\varphi}{2}}, \\ \cos\tfrac{1}{2}\varphi &= \pm\sqrt{\frac{1 + \cos\varphi}{2}}. \end{aligned}\right\} \tag{24}$$

The plus sign is to be used in the first formula of (24) when $\frac{1}{2}\varphi$ is a first-quadrant* or a second-quadrant angle, the minus sign when $\frac{1}{2}\varphi$ is a third-quadrant or a fourth-quadrant angle. The plus sign is to be used in the second equation of (24) when

* Occasionally it will be convenient to refer to an angle as belonging to a certain quadrant. If the initial ray of an angle extends from the origin along the positive x-axis, it is called a first-quadrant angle, a second-quadrant angle, a third-quadrant angle, or a fourth-quadrant angle according as its terminal side lies in the first, second, third, or fourth quadrant.

$\frac{1}{2}\varphi$ is a first-quadrant or a fourth-quadrant angle, the minus sign when $\frac{1}{2}\varphi$ is a second-quadrant or a third-quadrant angle.

To obtain a formula for $\tan \frac{1}{2}\varphi$, divide the first of equations (23) by the second to obtain

$$\tan \tfrac{1}{2}\varphi = \frac{\sin \frac{1}{2}\varphi}{\cos \frac{1}{2}\varphi} = \pm \sqrt{\frac{1 - \cos \varphi}{2}} \times \sqrt{\frac{2}{1 + \cos \varphi}},$$

or

$$\boldsymbol{\tan \tfrac{1}{2}\varphi = \pm \sqrt{\frac{1 - \cos \varphi}{1 + \cos \varphi}}}. \tag{25}$$

The plus sign is to be used when $\frac{1}{2}\varphi$ is a first-quadrant or a third-quadrant angle, the minus sign when $\frac{1}{2}\varphi$ is a second-quadrant or a fourth-quadrant angle. From (25) we also have

$$\tan \tfrac{1}{2}\varphi = \pm \sqrt{\frac{(1 - \cos \varphi)(1 - \cos \varphi)}{(1 + \cos \varphi)(1 - \cos \varphi)}} = \frac{1 - \cos \varphi}{\sin \varphi}. \tag{26}$$

Since $1 - \cos \varphi$ is never negative and $\sin \varphi$ always has the same sign as $\tan \frac{1}{2}\varphi$, the right-hand member of (26) does not require the $\pm$ sign.

EXERCISES 6-3

1. If $\sin \alpha = \frac{3}{5}$, $\cos \alpha = -\frac{4}{5}$, find $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, and $\tan \frac{1}{2}\alpha$.

2. Use formulas (24) to find $\sin 22\frac{1}{2}°$ and $\cos 22\frac{1}{2}°$ from the fact that $\cos 45° = 1/\sqrt{2}$.

3. Verify the following identities:

(*a*) $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.

(*b*) $\dfrac{\sin 2\alpha}{\sin \alpha} - \dfrac{\cos 2\alpha}{\cos \alpha} = \sec \alpha$.

(*c*) $\cos^2 (45° + x) - \sin^2 (45° + x) = -\sin 2x$.

(*d*) $\left(\sin \dfrac{\theta}{2} - \cos \dfrac{\theta}{2}\right)^2 = 1 - \sin \theta$.

(*e*) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$.

(*f*) $\dfrac{\sin 2\alpha + \sin \alpha}{1 + \cos \alpha + \cos 2\alpha} = \tan \alpha$.

(*g*) $\tan 2\theta = \dfrac{2}{\cot \theta - \tan \theta}$.

(*h*) $\tan \frac{1}{2}\varphi = \csc \varphi - \cot \varphi$.

4. Substitute $\theta = 2x, \varphi = x$ in $\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$ and then use the double-angle formulas to derive

$$\sin 3x = 3\sin x\cos^2 x - \sin^3 x = 3\sin x - 4\sin^3 x.$$

5. Using a method similar to the one suggested in Exercise 4, derive.

(*a*) $\cos 3x = 4\cos^3 x - 3\cos x$.
(*b*) $\sin 4x = 4\sin x\cos x(2\cos^2 x - 1)$.

6. Derive a formula expressing $\sin 4x$ in terms of $\sin x$ and a formula expressing $\tan 4x$ in terms of $\tan x$.

7. Prove that, if $z = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2z}{1+z^2}, \qquad \cos\theta = \frac{1-z^2}{1+z^2}, \qquad \tan\theta = \frac{2z}{1-z^2}.$$

8. Find $\sin 18°$ in radical form.

Hint. First write $\cos 3x = \sin 2x$ where $x = 18°$, and express both members in terms of $\sin x$ and $\cos x$. Solve the resulting equation for $\sin x$.

9. If θ is an angle in the second quadrant and $\tan\theta = -\frac{5}{12}$, find

(*a*) $\cot 2\theta$.
(*b*) $\cos(270° - 2\theta)$.
(*c*) $\sin(180° - \theta)$.
(*d*) $\csc(180° + 2\theta)$.

10. Show that

(*a*) $\cot\frac{x}{4} = \dfrac{\sin\frac{x}{2}}{1 - \cos\frac{x}{2}}$.

(*b*) $\cot\frac{x}{2} + \tan\frac{x}{2} = 2\csc x$.

(*c*) $\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \cos x$.

(*d*) $\tan\frac{1}{2}x = \dfrac{\sin x}{1 + \cos x}$.

(*e*) $\cot\frac{1}{2}x = \dfrac{\sin x}{1 - \cos x}$.

(*f*) $\sin 2x = \dfrac{2\cot x}{1 + \cot^2 x}$.

11. (*a*) Show that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

(*b*) Show that $\tan 4x = \dfrac{4\tan x(1 - \tan^2 x)}{1 - 6\tan^2 x + \tan^4 x}$.

12. In Fig. 6-7, AD bisects the angle A and DE is perpendicular to AB. Hence $DE = CD$. Show from the figure that

$$\tan \tfrac{1}{2}A = \frac{\sin A}{1 + \cos A}.$$

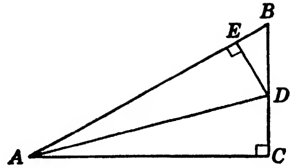
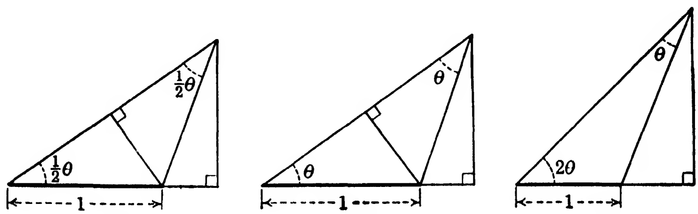


FIG. 6-7.

13. Find all line segments of these figures in terms of θ, and write several identities from your figures. Verify these identities in the usual way.



(a) (b) (c)

14. Prove the formula for $\tan(\alpha + \beta)$ from Fig. 6-8 by using line values.

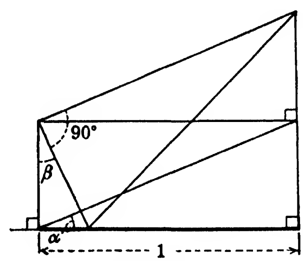


FIG. 6-8.

15. Prove that in a right triangle, C being the right angle, the following relations are true:

(a) $\sin 2A = \sin 2B.$

(b) $\tan 2A = \dfrac{2ab}{b^2 - a^2}.$

(c) $\cos 2A = \dfrac{b^2 - a^2}{c^2}.$

(d) $\cos 2A + \cos 2B = 0.$

(e) $\tan B = \cot A + \cos C.$

(f) $\sin 3A = \dfrac{3ab^2 - a^3}{c^3}.$

6-6. Conversion formulas. From (1) and (10), we have

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi,$$
$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi.$$

Adding these two formulas member by member, we get

$$\sin(\theta + \varphi) + \sin(\theta - \varphi) = 2 \sin\theta \cos\varphi. \tag{27}$$

Subtracting the second from the first, we obtain

$$\sin(\theta + \varphi) - \sin(\theta - \varphi) = 2 \cos\theta \sin\varphi. \tag{28}$$

From (1) and (11) we get

$$\cos(\theta + \varphi) = \cos\theta \cos\varphi - \sin\theta \sin\varphi,$$
$$\cos(\theta - \varphi) = \cos\theta \cos\varphi + \sin\theta \sin\varphi.$$

Adding these formulas member by member and afterwards subtracting the second from the first, we obtain

$$\cos(\theta + \varphi) + \cos(\theta - \varphi) = 2 \cos\theta \cos\varphi, \tag{29}$$
$$\cos(\theta + \varphi) - \cos(\theta - \varphi) = -2 \sin\theta \sin\varphi. \tag{30}$$

Formulas (27) to (30) should not be memorized but should be recalled by mentally carrying out their derivation from the addition formulas. These formulas are important because they enable us to express a product of sines and cosines as a sum of two or more expressions or to express a sum or a difference of two trigonometric functions in the form of a product. The following examples will illustrate the method of doing this.

Example 1. Express $\sin 5x - \sin 3x$ in the form of a product.

Solution. The left-hand member of (28) will be the desired difference if we set

$$\theta + \varphi = 5x, \qquad \theta - \varphi = 3x, \tag{a}$$

or, solving for θ and φ in terms of x,

$$\theta = 4x, \qquad \varphi = x. \tag{b}$$

Substituting θ and φ from (b) in (28), we obtain

$$\sin 5x - \sin 3x = 2 \cos 4x \sin x.$$

Example 2. Expand $\cos 2x \cos 3x \sin 4x$ into a sum of sines and cosines of multiple angles.

Solution. Using (29) with $\theta = 2x$, $\varphi = 3x$, we obtain

$$2 \cos 2x \cos 3x = \cos(2x + 3x) + \cos(2x - 3x),$$

or

$$2 \cos 2x \cos 3x = \cos 5x + \cos x. \tag{a}$$

Multiplying (*a*) through by $\sin 4x$ and dividing by 2, we get

$$\cos 2x \cos 3x \sin 4x = \tfrac{1}{2}(\cos 5x \sin 4x + \cos x \sin 4x). \qquad (b)$$

Then using (27) with $\theta = 4x$, $\varphi = 5x$, we obtain

$$2 \sin 4x \cos 5x = \sin (4x + 5x) + \sin (4x - 5x),$$

or

$$2 \sin 4x \cos 5x = \sin 9x - \sin x. \qquad (c)$$

Again using (27) with $\theta = 4x$, $\varphi = x$, we obtain

$$2 \cos x \sin 4x = \sin 5x + \sin 3x. \qquad (d)$$

Substituting $\sin 4x \cos 5x$ from (*c*) and $\cos x \sin 4x$ from (*d*) in (*b*), we obtain, after slight simplification,

$$\cos 2x \cos 3x \sin 4x = \tfrac{1}{4}(\sin 9x - \sin x + \sin 5x + \sin 3x).$$

A process similar to that carried out in (*a*) and (*b*) in Example 1 to find θ and φ in terms of the given angles may be used to derive another set of formulas that are convenient for transforming a sum to a product. Let

$$\theta + \varphi = \alpha, \qquad \theta - \varphi = \beta. \qquad (31)$$

Solving (31) simultaneously for θ and φ in terms of α and β, we get

$$\theta = \tfrac{1}{2}(\alpha + \beta), \qquad \varphi = \tfrac{1}{2}(\alpha - \beta). \qquad (32)$$

Replacing θ by $\frac{1}{2}(\alpha + \beta)$ and φ by $\frac{1}{2}(\alpha - \beta)$ in (27), (28), (29), and (30), we obtain

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta), \qquad (33)$$
$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta), \qquad (34)$$
$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta), \qquad (35)$$
$$\cos \alpha - \cos \beta = -2 \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta). \qquad (36)$$

EXERCISES 6-4

1. Express in the form of a product

(*a*) $\sin 35° + \sin 25°$.
(*b*) $\sin 45° - \sin 30°$.
(*c*) $\cos 65° + \cos 25°$.
(*d*) $\cos 75° - \cos 5°$.
(*e*) $\cos 4x + \cos 2x$.
(*f*) $\sin 5x - \sin 2x$.
(*g*) $\sin 3x + \sin x$.
(*h*) $\cos 5x - \cos 3x$.

2. Expand into a sum of sines and cosines of multiple angles:

(*a*) $\sin 3x \cos 7x$. (*b*) $\cos 3x \cos 7x$.
(*c*) $\sin x \sin 2x \cos 3x$. (*d*) $\cos 3x \cos 5x \sin 7x$.

Verify the following identities:

3. $\sin 32° + \sin 28° = \cos 2°$.

4. $\sin 50° - \sin 10° = \sqrt{3} \sin 20°$.

5. $\cos 80° - \cos 20° = -\sin 50°$.

6. $\cos 140° + \cos 100° + \cos 20° = 0$.

7. $\tan 50° + \cot 50° = 2 \sec 10°$.

8. $\cos 60° + \cos 30° = \sqrt{2} \cos 15°$.

9. $\sin 40° - \cos 70° = \sqrt{3} \sin 10°$.

10. $\sin (60° + \alpha) + \sin (60° - \alpha) = \sqrt{3} \cos \alpha$.

11. $\cos 5x + \cos 9x = 2 \cos 7x \cos 2x$.

12. $\dfrac{\sin 7x - \sin 5x}{\cos 7x + \cos 5x} = \tan x$.

13. $\dfrac{\sin 33° + \sin 3°}{\cos 33° + \cos 3°} = \tan 18°$.

14. $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \tan \frac{1}{2}(A - B) \cot \frac{1}{2}(A + B)$.

15. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A + B)$.

16. $\cos 20° - \sin 10° - \sin 50° = 0$.

17. $\sin (60° + x) - \sin x = \sin (60° - x)$.

18. $\cos (30° + y) - \cos (30° - y) = -\sin y$.

19. $\cos (x + 45°) + \cos (x - 45°) = \sqrt{2} \cos x$.

20. $\cos (Q + 45°) + \sin (Q - 45°) = 0$.

21. $\dfrac{\sin A + \sin B}{\cos A - \cos B} = -\cot \frac{1}{2}(A - B)$.

22. $\cos 3\alpha - \cos 7\alpha = 2 \sin 5\alpha \sin 2\alpha$.

23. $\dfrac{\sin 5x - \sin 2x}{\cos 2x - \cos 5x} = \cot \dfrac{7x}{2}$.

24. $\sin \theta + \sin 2\theta + \sin 3\theta = \sin 2\theta(1 + 2 \cos \theta)$.

25. $\cos \theta + \cos 2\theta + \cos 3\theta = \cos 2\theta(1 + 2 \cos \theta)$.

26. If $A + B + C = 180°$, prove that

(*a*) $\cos (A + B - C) = -\cos 2C$.

(*b*) $\sin A + \sin B - \sin C = 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \cos \dfrac{C}{2}$.

(*c*) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
(*d*) $\tan A - \cot B = \sec A \csc B \cos C$.

27. Prove $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \frac{1}{2}(\alpha - \beta)$.

MISCELLANEOUS EXERCISES 6-5

1. (*a*) Show that the value of $\sin 2\theta$ is less than the value of $2 \sin \theta$ for all values of θ between 0° and 90°.

(*b*) Show that the value of the fraction $\frac{\sin 2\theta}{2 \sin \theta}$ decreases from 1 to 0 as θ increases from 0° to 90°.

2. Given $\cot \alpha = \frac{4}{3}$ and $\cos \beta = -\frac{5}{13}$, find the value of each of the following if α and β each terminate in the third quadrant:

(*a*) $\cos (\alpha - \beta)$. (*b*) $\tan (\alpha + \beta)$. (*c*) $\sin (\beta - \alpha)$.
(*d*) $\cot (\alpha + \beta)$. (*e*) $\cot (\alpha - \beta)$. (*f*) $\tan (\beta - \alpha)$.

3. If $\cos \alpha = \frac{3}{5}$ and $\sin \beta = -\frac{3}{5}$, and if α is in the fourth and β in the third quadrant show that

(*a*) $\sin (\alpha + \beta) = +\frac{7}{25}$; $\cos (\alpha + \beta) = -\frac{24}{25}$; $\tan (\alpha + \beta) = -\frac{7}{24}$,
(*b*) $\sin (\alpha - \beta) = +1$; $\cos (\alpha - \beta) = 0$; $\tan (\alpha - \beta) = \infty$.

4. Prove that $\sin 180° = 0$ and $\cos 180° = -1$, using the functions of 120° and 60°.

5. Find $\tan (x + y)$ and $\tan (x - y)$, having given $\tan x = \frac{1}{2}$ and $\tan y = \frac{1}{4}$.

Verify each of the following:

6. $\tan (45° + x) = \dfrac{1 + \tan x}{1 - \tan x}$.

7. $\cot (y - 45°) = \dfrac{1 + \cot y}{1 - \cot y}$.

8. $\cot (B + 210°) = \dfrac{\sqrt{3} \cot B - 1}{\cot B + \sqrt{3}}$.

9. $\dfrac{\sin (x + y)}{\sin (x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

10. $\tan x + \tan y = \dfrac{\sin (x + y)}{\cos x \cos y}$.

11. $\dfrac{\tan (\theta - \phi) + \tan \phi}{1 - \tan (\theta - \phi) \tan \phi} = \tan \theta$.

12. $\tan (45° + x) - \tan (45° - x) = 2 \tan 2x$.

13. $\tan(45° + C) + \tan(45° - C) = 2 \sec 2C.$

14. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}.$

15. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}.$

16. $\dfrac{1 + \sin 2x}{1 - \sin 2x} = \left(\dfrac{\tan x + 1}{\tan x - 1}\right)^2.$

17. $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}.$

18. $\dfrac{\cos(x - y)}{\cos(x + y)} = \dfrac{1 + \tan x \tan y}{1 - \tan x \tan y}.$

19. $\tan A - \tan B = \dfrac{\sin(A - B)}{\cos A \cos B}.$

20. $\cot x + \cot y = \dfrac{\sin(x + y)}{\sin x \sin y}.$

21. $\cos(60° - A) = \dfrac{\cos A + \sqrt{3} \sin A}{2}.$

22. $\cos(x - 315°) = \dfrac{\cos x - \sin x}{\sqrt{2}}.$

23. $\cos 5\alpha \cos 4\alpha + \sin 5\alpha \sin 4\alpha = \cos \alpha.$

24. $\sin(x + 75°)\cos(x - 75°) - \cos(x + 75°)\sin(x - 75°) = \frac{1}{2}.$

25. $\cos(2x + y)\cos(x + 2y) + \sin(2x + y)\sin(x + 2y)$
$= \cos x \cos y + \sin x \sin y.$

26. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y.$

27. $\cos(x - y + z) = \cos x \cos y \cos z + \cos x \sin y \sin z$
$- \sin x \cos y \sin z + \sin x \sin y \cos z.$

28. $\sin(30° + x)\sin(30° - x) = \frac{1}{4}(\cos 2x - 2 \sin^2 x).$

29. $\sin(A + B)\sin(A - B) = \cos^2 B - \cos^2 A.$

30. $\left(\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)^2 = 1 + \sin x.$

31. $\dfrac{1 + \sec y}{\sec y} = 2 \cos^2 \dfrac{y}{2}.$

32. $2 \sin\left(45° + \dfrac{x - y}{2}\right) \cos\left(45° - \dfrac{x + y}{2}\right) = \cos y + \sin x.$

33. $1 + \tan x \tan \dfrac{x}{2} = \sec x.$

34. $\tan \dfrac{x}{2} + 2 \sin^2 \dfrac{x}{2} \cot x = \sin x.$

35. $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\tan\dfrac{\theta}{2}}{1-\tan\dfrac{\theta}{2}}.$

36. $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x} = \cot\dfrac{x}{2}.$

37. $1+\cot^2\dfrac{x}{2} = \dfrac{2}{\sin x\tan\dfrac{x}{2}}.$

38. $\dfrac{\tan^2\dfrac{x}{2}+\cot^2\dfrac{x}{2}}{\tan^2\dfrac{x}{2}-\cot^2\dfrac{x}{2}} = -\dfrac{1+\cos^2 x}{2\cos x}.$

39. Give the behavior of $\tan\dfrac{\theta}{2}+2\sin^2\dfrac{\theta}{2}\cot\theta$ as θ increases from $0°$ to $90°$.

40. Show that the value of $\tan^2\theta(1+\cos 2\theta)+2\cos^2\theta$ is the same for all values of θ.

41. Prove $\dfrac{\sin x+\cos x}{\cos x-\sin x} = \tan 2x+\sec 2x.$

42. Prove $\dfrac{\cot(90°+A)}{\cos 2A-1} = \csc 2A.$

43. Prove $\cos x\sin(y-z)+\cos y\sin(z-x)+\cos z\sin(x-y) = 0.$

44. Prove $\sin x\cos(y+z)-\sin y\cos(x+z) = \sin(x-y)\cos z.$

45. Prove $1-4\sin^2 x-2\sin^{x} x\cos 2x = \cos 2x.$

CHAPTER 7

SOLUTION OF THE OBLIQUE TRIANGLE

7-1. Law of sines. The object of this chapter is to develop important formulas that are useful in solving rectilinear figures and to indicate how they are applied.

In any triangle such as those of Fig. 7-1, A, B, and C represent the angles, and a, b, and c represent, respectively, the lengths of

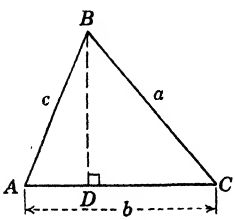


FIG. 7-1a.

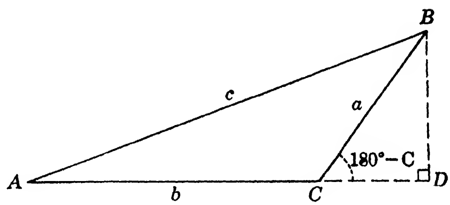


FIG. 7-1b.

the sides opposite these angles. Figure (a) represents a triangle all angles of which are acute; (b), a triangle containing an obtuse angle. In each figure the line DB is perpendicular to AC or AC produced. In either figure

$$\frac{DB}{c} = \sin A, \qquad \text{or} \qquad DB = c \sin A. \tag{1}$$

In (a), $DB/a = \sin C$, and in (b),

$$DB/a = \sin (180° - C) = \sin C.$$

In either case

$$DB = a \sin C. \tag{2}$$

Equating the value of DB from (1) to the value of DB from (2) and dividing the result by $\sin A \sin C$, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{3}$$

Similarly, by drawing a perpendicular from C to the opposite side of the triangle and reasoning as above, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{4}$$

Equations (3) and (4) may be combined in the equations

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{5}$$

The equations (5) are referred to as *the law of sines.* This law may be stated as follows: **The sides of a triangle are proportional to the sines of the opposite angles.**

Example. Express all line segments of Fig. 7-2(a) in terms of the given parts.

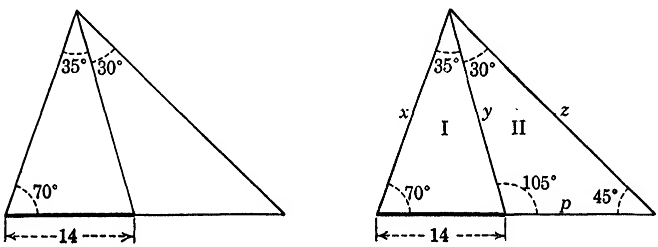


FIG. 7-2a. FIG. 7-2b.

Solution. Compute the angles and represent the unknown sides by letters; this gives us Fig. 7-2(b). Attending to triangle I, we think: x over sine of angle opposite (75°) equals 14 over sine of angle opposite (35°), and write

$$\frac{x}{\sin 75°} = \frac{14}{\sin 35°}, \quad \text{or} \quad x = \mathbf{14 \sin 75° \csc 35°}. \tag{a}$$

Again from triangle I, we write

$$\frac{y}{\sin 70°} = \frac{14}{\sin 35°}, \quad \text{or} \quad y = \mathbf{14 \sin 70° \csc 35°}. \tag{b}$$

From triangle II, we write

$$\frac{p}{\sin 30°} = \frac{y}{\sin 45°}, \qquad \frac{z}{\sin 105°} = \frac{y}{\sin 45°}, \tag{c}$$

or

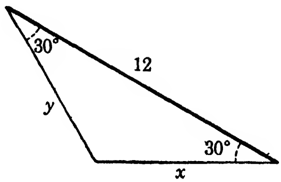
$$p = y\frac{\sin 30°}{\sin 45°}, \qquad z = y\frac{\sin 105°}{\sin 45°}. \tag{d}$$

Replacing y in (d) by its value from (b) and simplifying slightly, we obtain

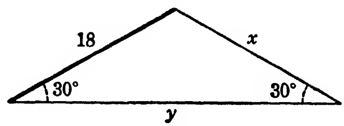
$$p = 14 \sin 70° \csc 35° \sin 30° \csc 45°.$$
$$z = 14 \sin 70° \csc 35° \sin 105° \csc 45°.$$

EXERCISES 7-1

1. Find x and y in radical form.

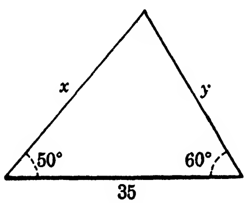


(a)

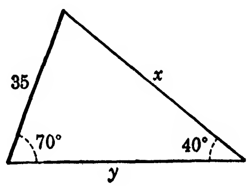


(b)

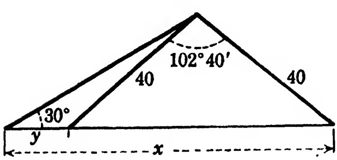
2. Express x and y in each of these figures in terms of the given parts.



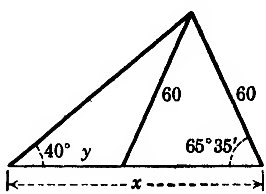
(a)



(b)



(c)



(d)

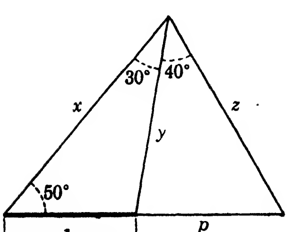


FIG. 7-3.

3. Find x, y, z, and p of Fig. 7-3 in terms of the given angles.

4. Find sin B where B is defined by Fig. 7-4. Also find the value of x in terms of B and the given parts.

5. Find the area of the triangle of Fig. 7-4 in terms of B and the given parts.

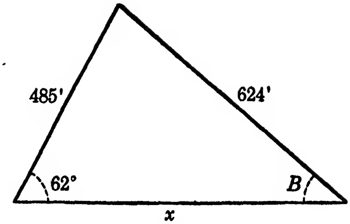


FIG. 7-4.

6. Express the lines x and y in Figs. 7-5 and 7-6 in terms of a and the given angles.

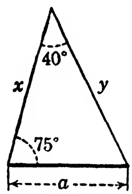


FIG. 7-5.

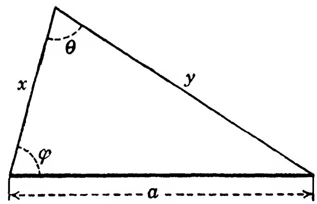


FIG. 7-6.

7. Express the lengths represented by x, y, z, and w of Fig. 7-7 in terms of the given parts.

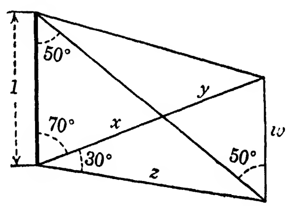


FIG. 7-7.

8. Use Fig. 7-8 to prove that

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

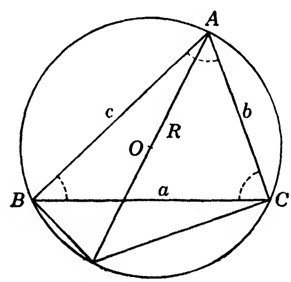


FIG. 7-8.

9. Show that $\sin (45° - \alpha) = \frac{2}{3} \sin 85°$ where α is defined by Fig. 7-9.

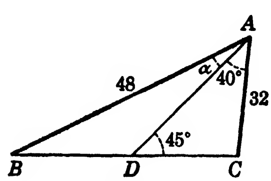


FIG. 7-9.

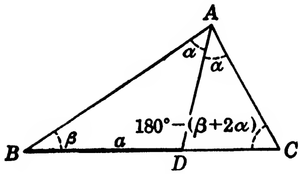


FIG. 7-10.

10. Express all segments in Fig. 7-10 in terms of a, α, and β and then show that $\frac{BD}{DC} = \frac{BA}{AC}$.

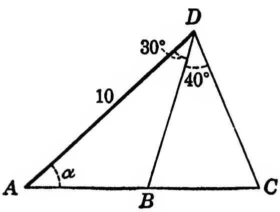


FIG. 7-11.

11. If $AB = BC$ in Fig. 7-11, prove that

$$\cot \alpha = \frac{\sin 40^\circ - \sin 30^\circ \cos 70^\circ}{\sin 30^\circ \sin 70^\circ}.$$

7-2. Application of the law of sines to the solution of oblique triangles. If the values given in a problem are small and easily handled by simple arithmetic, the solution with the law of sines can be done by means of multiplication and division. However, the law of sines does lend itself readily to the use of logarithms. The student should now recall the forms and the general method of procedure used in the solution of right triangles by logarithms. A similar method will be used with oblique triangles. It may be summarized as follows:

a. Draw a figure of the triangle to be solved, lettering it in the conventional way. Encircle the given parts.

b. Write the formulas to be used in the solution.

c. Make a complete form for the computation before looking up any logarithms.

d. Fill in the form.

7-3. Solve a triangle, given one side and two angles. The law of sines involves four variables: two sides and two angles. If three of these are known, the fourth one can be computed. From a study of the formula, one can readily see that it is to be used only if the known values are one side and two angles or two sides and one angle which must necessarily be opposite one of the sides. Recall also that, if two angles of a triangle are given, the third angle is determined.

Example. Given $a = 24.31$, $A = 45°18'$, and $B = 22°11'$ (see Fig. 7-12). Find b, c, and C.

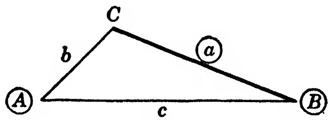


FIG. 7-12.

Solution. Since $A + B + C = 180°$,

$$C = 180° - (45°18' + 22°11') = 112°31'.$$

To find b, choose the formula from the law of sines which contains b and three known parts. Solve this formula for b, to obtain

$$b = \frac{a \sin B}{\sin A}. \tag{a}$$

Similarly,

$$c = \frac{a \sin C}{\sin A}. \tag{b}$$

(a) $b = \dfrac{a \sin B}{\sin A} = \dfrac{24.31 \sin 22°11'}{\sin 45°18'}$.

$$\begin{aligned} \log 24.31 &= 1.3857 \\ \log \sin 22°11' &= 9.5770 - 10 \\ \text{colog} \sin 45°18' &= 0.1483 \\ \hline \log b &= 1.1110 \\ \therefore b &= \mathbf{12.91.} \end{aligned}$$

(b) $c = \dfrac{a \sin C}{\sin A} = \dfrac{24.31 \sin 112°31'}{\sin 45°18'}$.

$$\begin{aligned} \log 24.31 &= 1.3857 \\ \log \sin 112°31' &= 9.9656 - 10 \\ \text{colog} \sin 45°18' &= 0.1483 \\ \hline \log c &= 1.4996 \\ \therefore c &= \mathbf{31.59.} \end{aligned}$$

The following compact form is recommended. The letter in parentheses above each column refers to the formula associated with the column.

	(a)	(b)
$a = 24.31$	$\log a = 1.3857$	$\log a = 1.3857$
$A = 45°18'$	$\text{col sin } A = 0.1483$	$\text{col sin } A = 0.1483$
$B = 22°11'$	*$\text{l sin } B = 9.5770 - 10$	
$b = \mathbf{12.91}$	$\log b = 1.1110$	
$C = \mathbf{112°31'}$		$\text{l sin } C = 9.9656 - 10$
$c = \mathbf{31.59}$		$\log c = 1.4996$

Check. The solution may be checked by using a form of the law of sines that was not used in the solution.

Does $\dfrac{12.91}{\sin 22°11'}$ equal $\dfrac{24.31}{\sin 45°18'}$?

$$\begin{aligned} \log 12.91 &= 11.1110 - 10 \\ \text{l sin } 22°11' &= \underline{\ 9.5770 - 10\ } \\ &\ 1.5340 \end{aligned} \qquad \begin{aligned} \log 24.31 &= 11.3857 - 10 \\ \text{l sin } 45°18' &= \underline{\ 9.8517 - 10\ } \\ &\ 1.5340 \end{aligned}$$

7-4. Use of the slide rule with the law of sines. For convenience of reference we repeat here the slide-rule setting for applying the law of sines to solve a triangle:

Rule A. To apply the law of sines for solving a triangle,
push the hairline to any known side on *D*,
draw under the hairline the opposite known angle on *S*;
push the hairline to any other side on *D*,
read at the hairline the angle opposite on *S*;
push the hairline to any other known angle on *S*,
read at the hairline the side opposite on *D*.

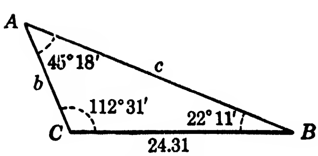


FIG. 7-13.

To solve the triangle in Art. 7-3 by means of the slide rule, we first find $C = 112°31'$ from the relation $A + B + C = 180°$ and then use Rule A. Hence, construct the triangle shown in Fig. 7-13, and

push hairline to 24.31 on D,
draw 45°18′ of S under the hairline,
push hairline to 22°11′ on S,
at the hairline read b = **12.91,**

* Note that "l" is used in these forms to abbreviate the word log and "col" to abbreviate colog.

push hairline to $67°29'$ ($= 180° - 112°31'$) on S,
at the hairline read $c =$ **31.6.**

EXERCISES 7-2

Solve the following triangles:

1. $A = 54°28'$, $B = 103°8'$, $a = 3.695$.	**2.** $B = 38°13'$, $C = 60°$, $a = 7013$.	**3.** $A = 64°56'$, $B = 47°29'$, $c = 913.4$.
4. $A = 47°23'$, $C = 70°17'$, $c = 227.2$.	**5.** $A = 71°14'$, $B = 40°34'$, $c = 236.5$.	**6.** $A = 25°33'$, $B = 133°13'$, $a = 411.4$.

7. A line AB along one bank of a stream is 562 ft. long, and C is a point on the opposite bank. The angle BAC is $53°18'$, and the angle ABC is $48°36'$. Find the width of the stream.

8. A vertical plane contains a 132-ft. hillside tunnel sloping downward at 14° with the horizontal and cuts the hillside in a line sloping upwards at 18°. What is the vertical distance from the bottom of the tunnel to the surface of the hill?

9. Prove that the area K of triangle ABC in Fig. 7-14 is given by

$$K = \frac{b^2 \sin A \sin C}{2 \sin (A + C)}.$$

Hint. First find c in terms of encircled parts; then find h and use the formula $K = \frac{1}{2}ch$.

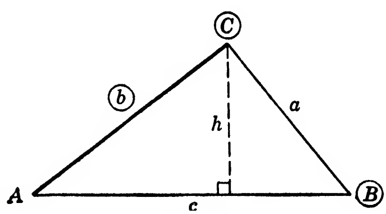


FIG. 7-14.

10. Use the formula in Exercise 9 to find the area of the triangle in (*a*) Exercise 1; (*b*) Exercise 6.

11. A shore station at point A is 5280 ft. from another at point B. Find the distance from each of the shore stations to an enemy ship at point C if angle ABC is $83°37'$ and angle BAC is $85°1'$.

12. A surveyor running a line due east reached the edge of a swamp. He then ran a line 2000 ft. in a direction S. 47° E., and from the point thus reached he ran a line in the direction N. 52° 20′ E. How far had he continued on this latter line when he reached a point on the original line extended?

13. A building 75.2 ft. high stands at the upper end of a street that slopes down at an angle of $6°52'$ with the horizontal. How far down the

street from the building is a point at which the angle of elevation of the top of the building is 13°58′?

14. From the top of a hill the angles of depression of the top and the bottom of a building 42.5 ft. high are observed to be 36° and 43°, respectively. Find the height of the hill if the building is at the foot of the hill.

7-5. Solve a triangle, given two sides and the angle opposite one of them. In this case, as in Art. 7-3, the triangle is solved by means of the law of sines and the relation $A + B + C = 180°$. However, this case needs further discussion, for in one instance an ambiguity exists.

Ambiguous case. When the side opposite the given angle is less than the other given side, there are three possibilities: no solution, one solution, or two solutions. Let us investigate the situation in detail.

Let A, a, and b of Figs. 7-15, 7-16, 7-17 be the given parts in which $a < b$. The perpendicular from C to side c is $b \sin A$.

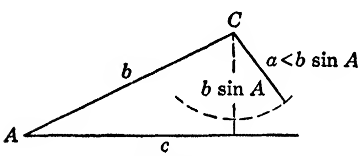


FIG. 7-15.

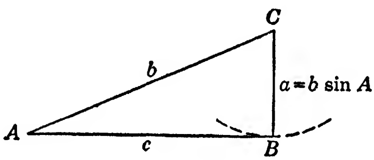


FIG. 7-16.

a. If, in Fig. 7-15, $a < b \sin A$, side a is too short to reach side c. Hence there is no solution.

b. If, in Fig. 7-16, $a = b \sin A$, side a just reaches side c. Hence there is one solution, a right triangle.

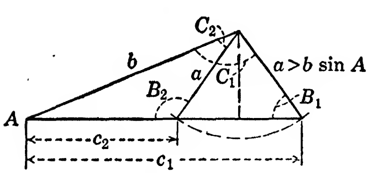


FIG. 7-17.

c. If, in Fig. 7-17, $a > b \sin A$, there are two solutions. In practice this is the most probable condition. Notice that B_1 and B_2 are supplementary angles.

These results may be summarized thus: If in triangle ABC, $a < b$, we have no solution when $a < b \sin A$; one solution when $a = b \sin A$; two solutions when $a > b \sin A$.

In the ambiguous case it is not necessary to determine the number of solutions in the foregoing manner before proceeding to solve the triangle, for we shall discover the nature of the

situation as soon as we have added the first column of logarithms in the solution. Hence proceed with the computation, and when log sin B has been found observe that

(*a*) **if log sin $B > 0$, then sin $B > 1$, and there is no solution;**
(*b*) **if log sin $B = 0$, then sin $B = 1$ and there is one solution, a right triangle;**
(*c*) **if log sin $B < 0$, then sin $B < 1$, and there are two solutions.**

Hence the procedure is as follows:

a. Determine whether the ambiguous case exists by noting whether the side opposite the given angle is less than the side adjacent to the given angle ($a < b$).

b. Proceed with the computation and if the ambiguous case is involved expect two solutions, but keep in mind that there may be no solution or one solution.

Example 1. Given $a = 67.53$, $b = 56.83$, and $A = 79°15'$. Find c, B, and C.

Solution. By inspection it is observed that $a > b$. Hence this is not the ambiguous case.

To find B, from the law of sines choose the formula containing B and the three known parts. Solve this formula for B to obtain

$$\sin B = \frac{b \sin A}{a}. \qquad (a)$$

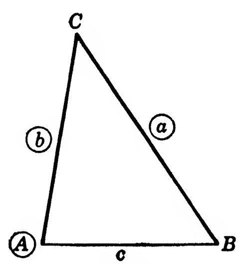


FIG. 7-18.

After finding B from (a), determine C from the relation

$$A + B + C = 180°.$$

Then write the law of sines involving c, C, and the knowns a and A to obtain

$$c = \frac{a \sin C}{\sin A}. \qquad (b)$$

The solution is displayed in the following form. The letter in parentheses above each column refers to the formula associated with the column.

	(a)	(b)
$b = 56.83$	$\log b = 1.7545$	
$A = 79°15'$	$\text{l} \sin A = 9.9923 - 10$	$\text{col} \sin A = 0.0077$
$a = 67.53$	$\text{col}\ a = 8.1705 - 10$	$\log a = 1.8295$
$B = \mathbf{55°45'}$	$\text{l} \sin B = 9.9173 - 10$	
$C = \mathbf{45°0'}$		$\text{l} \sin C = 9.8495 - 10$
$c = \mathbf{48.61}$		$\log c = 1.6867$

Check. Does $\dfrac{c}{\sin C}$ equal $\dfrac{b}{\sin B}$?

$$\begin{array}{rl} \log c = & 11.6867 - 10 \\ \text{l} \sin C = & \underline{\ \ 9.8495 - 10} \\ & 1.8372 \end{array} \qquad \begin{array}{rl} \log b = & 11.7545 - 10 \\ \text{l} \sin B = & \underline{\ \ 9.9173 - 10} \\ & 1.8372 \end{array}$$

To solve Example 1 by means of the slide rule, set the proportion

$$\frac{67.5}{\sin 79°15'} = \frac{56.8}{\sin B} = \frac{c}{\sin C}$$

on the rule, and read $B = \mathbf{55°45'}$. From the relation

$$A + B + C = 180°,$$

get $C = \mathbf{45°}$; then on the slide rule read $c = \mathbf{48.6}$.

Example 2. Given $a = 9.467$, $b = 14.43$, and $A = 11°14'$. Find c, B, and C.

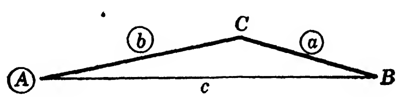


FIG. 7-19.

Solution. By inspection it is observed that $a > b$. Hence this is the ambiguous case. When log sin B has been computed, we shall determine the number of solutions. The formulas, obtained as in Example 1, are

$$\sin B = \frac{b \sin A}{a}, \tag{a}$$

$$C = 180° - (A + B),$$

$$c = \frac{a \sin C}{\sin A}. \tag{b}$$

The solution is displayed in the following form

	(a)	(b)	(b)
$b = 14.43$	$\log b = 1.1593$		
$A = 11°14'$	$\text{l sin } A = 9.2896 - 10$	$\text{col sin } A = 0.7104$	$\text{col sin } A = 0.7104$
$a = 9.467$	$\text{col } a = 9.0237 - 10$	$\log a = 0.9763$	$\log a = 0.9763$
$B_1 = \mathbf{17°16'}$	$\text{l sin } B_1 = 9.4726 - 10$		
$B_2 = \mathbf{162°44'}$			
$C_1 = \mathbf{151°30'}$		$\text{l sin } C_1 = 9.6787 - 10$	
$C_2 = \mathbf{6°2'}$			$\text{l sin } C_2 = 9.0216 - 10$
$c_1 = \mathbf{23.19}$		$\log c_1 = 1.3654$	
$c_2 = \mathbf{5.108}$			$\log c_2 = 0.7083$

Since log sin B from the first column was found to be negative, we concluded that there were two solutions. Since sin B is positive in both the first and the second quadrants, we obtained two supplementary angles B_1 and B_2 from log sin B.

Check. Does $\dfrac{b}{\sin B_1}$ equal $\dfrac{c_1}{\sin C_1}$?

$$\begin{aligned} \log b &= 11.1593 - 10 \\ \text{l sin } B_1 &= 9.4726 - 10 \\ \hline &\ 1.6867 \end{aligned} \qquad \begin{aligned} \log c_1 &= 11.3654 - 10 \\ \text{l sin } C_1 &= 9.6787 - 10 \\ \hline &\ 1.6867 \end{aligned}$$

Check. Does $\dfrac{b}{\sin B_2}$ equal $\dfrac{c_2}{\sin C_2}$?

$$\begin{aligned} \log b &= 11.1593 - 10 \\ \text{l sin } B_2 &= 9.4726 - 10 \\ \hline &\ 1.6867 \end{aligned} \qquad \begin{aligned} \log c_2 &= 10.7083 - 10 \\ \text{l sin } C_2 &= 9.0216 - 10 \\ \hline &\ 1.6867 \end{aligned}$$

To solve the triangle of Example 2 by means of the slide rule, use the same general line of argument applied in the log-

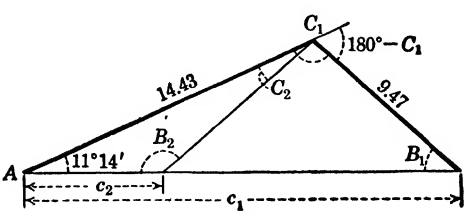


FIG. 7-20.

arithmic solution, but employ Rule A of Art. 7-4 for the computation. Hence draw Fig. 7-20 and

push hairline to 947 on D,
draw 11°14′ of S under hairline,

push hairline to 14.43 on D,*
at the hairline read $B_1 = $ **17°17′** on S;
push hairline to $180° - C_1 = 28°31'$ on S,
at the hairline read $c_1 = $ **23.2** on D;
compute $C_2 = B_1 - 11°14' = $ **6°3′**,
push hairline to 6°3′ on S,
at the hairline read $c_2 = $ **5.12** on D.

EXERCISES 7-3

Solve the following triangles:

1. $a = 309$, $b = 360$, $A = 21°14'$.

2. $b = 316$, $c = 360$, $B = 21°17'$.

3. $A = 41°13'$, $a = 77.04$, $b = 91.06$.

4. $b = 115.9$, $c = 139.1$, $B = 43°12'$.

5. $a = 294$, $b = 189$, $A = 67°32'$.

6. $b = 71.82$, $c = 78.49$, $B = 66°12'$.

7. $a = 48.13$, $b = 35.83$, $A = 36°24'$.

8. $a = 32.24$, $b = 50.21$, $A = 32°19'$.

9. $a = 4236$, $b = 5.123$, $A = 54°18'$.

10. $b = 216.5$, $c = 177.1$, $C = 35°36'$.

11. $a = 341.9$, $b = 745.9$, $A = 43°36'$.

12. $a = 95.21$, $b = 126.4$, $A = 51°41'$.

13. It is desired to measure the distance AB between two points on opposite sides of a lake. A point C, easily accessible to both A and B, is chosen. It is found that $AC = 8461$ and $BC = 10{,}246$. At A the angle BAC is found to be 26°33′. Find the distance AB.

14. Two wires are run from the same point on the vertical edge of a building to a level courtyard below. One wire is 42.45 ft. long and

* Occasionally it will be necessary to use the following rule: when a number is to be read on the D scale opposite a number on the slide and cannot be read because the slide projects beyond the body of the rule, push the hairline to the index of the C scale inside the body and draw the other index of the C scale under the hairline. The desired reading can then be made.

makes an angle of 58° with the horizontal. The other wire is 48.60 ft. long and lies in the same vertical plane with the first but on the opposite side of the edge. Find the inclination of the second wire to the yard and the distance between anchor points.

15. The distance from a point A to a point C cannot be measured directly but is estimated to be about $\frac{1}{4}$ mile. From a point B, $BA =$ 7201 ft., and $BC = 6180$ ft. Angle BAC is found to be 41°14′. Find the distance AC.

7-6. The law of tangents. Mollweide's equations. The equations referred to in the title of this article are easily deduced from the law of sines. The law of tangents, the proof of which follows directly, is used to solve a triangle when two sides and the included angle are given. Mollweide's equations are excellent equations for checking purposes.

From the law of sines, we have

$$\frac{a}{b} = \frac{\sin A}{\sin B}. \tag{6}$$

Subtracting 1 from each side of (6), we have

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1, \qquad \text{or} \qquad \frac{a - b}{b} = \frac{\sin A - \sin B}{\sin B}. \tag{7}$$

Adding 1 to each side of (6), we have

$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1, \qquad \text{or} \qquad \frac{a + b}{b} = \frac{\sin A + \sin B}{\sin B}. \tag{8}$$

By dividing (7) and (8) member by member, we obtain

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

Transforming the right-hand member of this equation by means of the formulas of Art. 6-6, we obtain

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}.$$

The right-hand member reduces to

$$\tan \tfrac{1}{2}(A - B) \div \tan \tfrac{1}{2}(A + B).$$

$$\therefore \boldsymbol{\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}}. \tag{9}$$

Another formula may be obtained by replacing a by c and A by C in (9) and a third, by replacing b by c and B by C in (9).

When $b > a$, both sides of (9) are negative. In this case it is convenient to write the formula in the form

$$\frac{b-a}{b+a} = \frac{\tan \frac{1}{2}(B-A)}{\tan \frac{1}{2}(B+A)}, \tag{10}$$

so that both members are positive.

The formulas often called Mollweide's equations are derived as follows:

From the law of sines, we have

$$\frac{a}{c} = \frac{\sin A}{\sin C}, \qquad \text{and} \qquad \frac{b}{c} = \frac{\sin B}{\sin C}. \tag{11}$$

Adding equations (11) member by member, we obtain

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}. \tag{12}$$

Transforming the right-hand member of this equation by means of formula (18) of Art. 6-5 and formula (33) of Art. 6-6, we obtain

$$\frac{a+b}{c} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}. \tag{13}$$

Since $A + B = 180° - C$,

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}(180° - C) = \cos \tfrac{1}{2}C.$$

Hence Mollweide's first equation may be written in the form

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}. \tag{14}$$

Mollweide's second equation,

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}, \tag{15}$$

is derived in a similar manner.

7-7. Solution of an oblique triangle by means of the law of tangents. Given two sides and the included angle. With these

given parts, the triangle can be solved by means of the law of tangents and the law of sines. The law of tangents gives the angles opposite the given sides, and the law of sines can then be used to find the third side. The result may be checked by means of Mollweide's equations.

Example 1. Given $c = 1.039$, $a = 6.752$, and $B = 127^\circ 9$. Find A, C, and b.

Solution. From the relation $A + B + C = 180^\circ$, we have $A + C = 180^\circ - B$, or

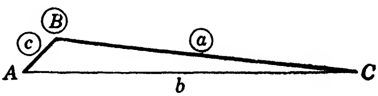


FIG. 7-21.

$$\tfrac{1}{2}(A + C) = \tfrac{1}{2}(180^\circ - 127^\circ 9') = 26^\circ 25.5'.$$

From the law of tangents, we have

$$\tan \tfrac{1}{2}(A - C) = \frac{(a - c)}{(a + c)} \tan \tfrac{1}{2}(A + C), \qquad (a)$$

and from the law of sines

$$b = \frac{a \sin B}{\sin A}. \qquad (b)$$

$$(a)\ \tan \frac{1}{2}(A - C) = \frac{5.713}{7.791} \tan 26^\circ 25.5'.$$

$$\begin{aligned}
\log 5.713 &= 0.7568 \\
\log \tan 26^\circ 25.5' &= 9.6963 - 10 \\
\operatorname{colog} 7.791 &= \underline{9.1084 - 10} \\
\log \tan \tfrac{1}{2}(A - C) &= 9.5615 - 10 \\
\therefore \tfrac{1}{2}(A - C) &= 20^\circ 1. \\
\text{But,} \qquad \tfrac{1}{2}(A + C) &= 26^\circ 25.5' \\
\therefore A &= \mathbf{46^\circ 26.5'}, \\
\text{and} \qquad C &= \mathbf{6^\circ 24.5'}.
\end{aligned}$$

$$(b)\ b = \frac{a \sin B}{\sin A} = \frac{6.752 \sin 127^\circ 9'}{\sin 46^\circ 26.5'}.$$

$$\begin{aligned}
\log 6.752 &= 0.8294 \\
\log \sin 127^\circ 9' &= 9.9015 - 10 \\
\operatorname{colog} \sin 46^\circ 26.5' &= \underline{0.1398} \\
\log b &= 0.8707 \\
\therefore b &= \mathbf{7.425}.
\end{aligned}$$

The solution is displayed in the following compact form:

	(a)	(b)
$B = 127°9'$		$\text{l} \sin B = 9.9015 - 10$
$a = 6.752$		$\log a = 0.8294$
$c = 1.039$		
$a - c = 5.713$	$\log (a - c) = 0.7568$	
$a + c = 7.791$	$\text{colog}\ (a + c) = 9.1084 - 10$	
$\frac{1}{2}(A + C) = 26°25.5'$	$\text{l} \tan \frac{1}{2}(A + C) = 9.6963 - 10$	
$\frac{1}{2}(A - C) = 20°1'$	$\text{l} \tan \frac{1}{2}(A - C) = 9.5615 - 10$	
$A = \mathbf{46°26.5'}$		$\text{col} \sin A = 0.1398$
$C = \mathbf{6°24.5'}$		
$b = \mathbf{7.425}$		$\log b = 0.8707$

The solution may be checked by one of Molweide's equations. Thus,

$$\text{Does } \frac{a - c}{b} \text{ equal } \frac{\sin \frac{1}{2}(A - C)}{\cos \frac{1}{2}B}?$$

$$\begin{aligned} \log (a - c) &= 10.7568 - 10 \\ \log b &= 0.8707 \\ \hline &9.8861 - 10 \end{aligned} \qquad \begin{aligned} \text{l} \sin \tfrac{1}{2}(A - C) &= 19.5344 - 20 \\ \text{l} \cos \tfrac{1}{2}B &= 9.6484 - 10 \\ \hline &9.8860 - 10 \end{aligned}$$

The difference of 0.0001 between the two results may be disregarded.

The following solution will illustrate the method of using the slide rule to solve a triangle when two of its sides and the included angle are known:

Example 2. Solve the triangle in which $b = 28.7$, $c = 45.2$, $A = 47°$.

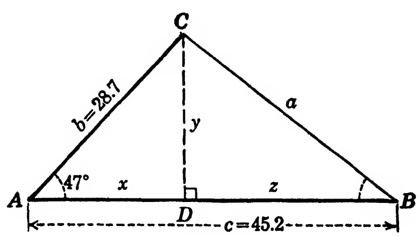


FIG. 7-22.

Solution. In Fig. 7-22 draw line CD perpendicular to AB, and solve the right triangle ACD. Knowing x, get $z = 45.2 - x$.

Then, knowing the two legs y and z of right triangle DBC, solve it by the method of Art. 14-18. This leads to the following settings:

set right index of C to 28.7 on D,
opposite 43° on S read $x = 19.6$ on D,
opposite 47° on S read $y = 21$ on D;
compute $z = 45.2 - 19.6 = 25.6$,
set right index of C to 25.6 on D,
push hairline to 21 on D,
at hairline read $B =$ **39°22′** on T;
draw 39°22′ of S under the hairline,
opposite index of C read $a =$ **33.1** on D.
Evidently angle $C = 43° + 90° - 39°22' =$ **93°38′.**

EXERCISES 7-4

Solve the following triangles:

1. $a = 17$, $b = 12$, $C = 59°17'$.

2. $a = 748$, $b = 375$, $C = 63°36'$.

3. $b = 232.2$, $c = 195.6$, $A = 61°13'$.

4. $a = 27.92$, $b = 42.38$, $C = 39°40'$.

5. $b = 85.25$, $c = 105.6$, $A = 50°40'$.

6. $a = 0.5931$, $b = 0.2273$, $C = 64°38'$.

7. $a = 6.239$, $b = 2.348$, $C = 110°32'$.

8. $a = 35.24$, $b = 18.48$, $C = 110°41'$.

9. The end A of a boom AB is attached to the platform of a crane and a cable BC connects the end B to a point C on top of the crane (see Fig. 7-23). If $AB = 35$ ft., $AC = 15$ ft., and angle $CAB = 95°$, find the length of the cable.

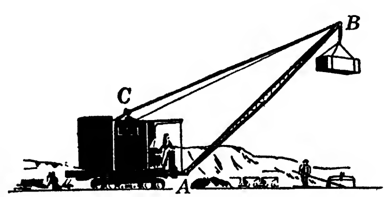


FIG. 7-23.

10. From a point 5890 ft. from one end of a lake and 6728 ft. from the other end, the lake subtends an angle of 47°18′. Find the length of the lake.

11. A triangular tract of land is to be enclosed by a fence. The side $AB = 54.23$ ft.; side $CB = 29.48$ ft.; the included angle B is $95°40'$. Find the amount of fencing needed to enclose the triangular plot.

12. From the top of a lighthouse 188.6 ft. above sea level, the angle of depression of a ship was $5°30'$, and its compass bearing was $16°48'$. One hour later the angle of depression was $4°10'$ and the compass bearing, $143°4'$. Find the distance traveled by the ship and its compass course.

13. Two yachts start from the same place at the same time. Yacht A sails at 10 knots on compass course 62°. Yacht B sails at 8 knots on compass course 135°. How far apart are they at the end of 40 min., and what is the bearing of yacht B from yacht A?

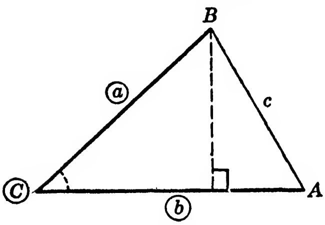


FIG. 7-24.

14. Prove that the area K of the triangle shown in Fig. 7-24 is given by

$$K = \tfrac{1}{2}ab \sin C.$$

Use the formula just derived to find the area of the triangle of (a) Exercise 1; (b) Exercise 7.

15. From a mountain peak in a vertical plane through a straight tunnel, the angles of depression of its ends are $42°41'$ and $52°22'$, and the corresponding distances from the peak to the ends of the tunnel are 3710 ft. and 4100 ft., respectively. Find the length of the tunnel.

16. From a ship two lighthouses bear N. 40° E. After the ship has sailed 15 miles on a course of 135°, they bear 10° and 345°, respectively. Find the distance between them and the distance from the ship in the latter position to the more distant lighthouse.

17. Two men, A and B, start at the same point on the circumference of a circle of radius 900 ft. and walk at the rate of 350 ft. per minute. If A walks toward the center of the circle and B walks along the circumference, find how far apart the two men are at the end of 1 min.

7-8. The law of cosines. In the triangles of Fig. 7-25 denote the angles by A, B, and C, and the sides opposite these angles by a, b, and c, respectively. Draw the perpendicular p from one of the vertices C of the triangle to the opposite side c in (a), or c produced in (b). In either figure

$$AD = b \cos A. \tag{16}$$

In (a)

$$DB = c - AD = c - b \cos A,$$

and in (*b*)

$$BD = AD - AB = b \cos A - c. \tag{17}$$

Since $(c - b \cos A)^2 = (b \cos A - c)^2$, we have for each triangle

$$b^2 - b^2 \cos^2 A = p^2 = a^2 - (c - b \cos A)^2.$$

Simplifying and solving for a^2, we obtain

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A.} \tag{18}$$

Similarly, by drawing perpendiculars from A and B to the opposite sides or the opposite sides produced, we obtain

$$\left.\begin{aligned} \boldsymbol{b^2} &= \boldsymbol{a^2 + c^2 - 2ac \cos B}, \\ \boldsymbol{c^2} &= \boldsymbol{a^2 + b^2 - 2ab \cos C}. \end{aligned}\right\} \tag{19}$$

The law of cosines embodied in equations (18) and (19) may be stated as follows: **The square of any side of a plane triangle**

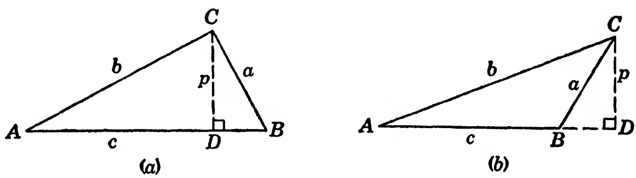


FIG. 7-25.

is equal to the sum of the squares of the other two sides diminished by twice the product of those two sides and the cosine of their included angle.

The law of cosines does not lend itself to the use of logarithms so readily as the law of sines and the law of tangents. It can be used very easily when the numbers involved can be handled conveniently by ordinary arithmetic. However, each term of the formula, that is, a^2 alone and c^2 alone and $2ac \cos B$ alone, can be evaluated by logarithms.

EXERCISES 7-5

1. Solve the law of cosines for (*a*) $\cos A$, (*b*) $\cos B$, (*c*) $\cos C$.

2. Find the third side of a triangle in which

(*a*) Two sides are 5 and 8 and the included angle is 60°.
(*b*) Two sides are 15 and 24 and the included angle is 50°30′.
(*c*) Two sides are 18 and 10 and the included angle is 22°16′.
(*d*) Two sides are 5.50 and 4.25 and the included angle is 34°28′.
(*e*) Two sides are 155.9 and 167.8 and the included angle is 49°24′.
(*f*) Two sides are 2.5 and 4 and the included angle is 120°.
(*g*) Two sides are 7.5 and 12 and the included angle is 110°35′.

3. Find the three angles in a triangle in which the sides are

(*a*) 16, 20, 25. (*b*) 5, 6, 7.
(*c*) 8, 4, 6. (*d*) 80, 100, 120.

4. Show that the law of cosines reduces to the Pythagorean theorem, if it is used to find the hypotenuse of a right triangle when the two legs are given.

5. The diagonals of a parallelogram are 16 and 24 and one of the angles that they form is 35°28′. Find the sides of the parallelogram.

6. If the three sides of an oblique triangle are a, b, and c, show that the sum of the squares of the sides equals

$$2(ab \cos C + bc \cos A + ca \cos B).$$

7-9. The half-angle formulas. Although the law of cosines may be used to solve a triangle when the three sides are given, it is not convenient to use in logarithmic computation. We shall now derive from the law of cosines other formulas that are well adapted to logarithmic computation.

From the first equation of (24) Art. 6-5, we obtain

$$2 \sin^2 \tfrac{1}{2}A = 1 - \cos A, \tag{20}$$

and from the law of cosines, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \tag{21}$$

Substituting the value of cos A from (21) in (20), we get

$$\begin{aligned} 2 \sin^2 \tfrac{1}{2}A &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\ &= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} \\ &= \frac{(a + b - c)(a - b + c)}{2bc}. \end{aligned} \tag{22}$$

Let

$$a + b + c = 2s. \tag{23}$$

Subtracting $2a$, $2b$, and $2c$ from each member of (23) we obtain, respectively,

$$\begin{aligned} -a + b + c &= 2(s - a), \\ a - b + c &= 2(s - b), \\ a + b - c &= 2(s - c). \end{aligned}$$

Substituting from the last two of these equations in (22) and simplifying slightly, we get

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}. \tag{24}$$

Similarly,

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s - c)(s - a)}{ca}}, \tag{25}$$

and

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s - a)(s - b)}{ab}}. \tag{26}$$

From the second equation of (24) Art. 6-5 and (21), we obtain

$$\begin{aligned} 2 \cos^2 \tfrac{1}{2}A &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b + c)^2 - a^2}{2bc} \\ &= \frac{(a + b + c)(-a + b + c)}{2bc} \\ &= \frac{(2s)2(s - a)}{2bc}. \end{aligned}$$

Hence

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}}. \tag{27}$$

Similarly,

$$\cos \tfrac{1}{2}B = \sqrt{\frac{s(s - b)}{ca}}, \tag{28}$$

and

$$\cos \tfrac{1}{2}C = \sqrt{\frac{s(s - c)}{ab}}. \tag{29}$$

Since $\tan \frac{1}{2}A = \dfrac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A}$, we get by substitution from (24) and (27)

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{30}$$

Similarly,

$$\tan \tfrac{1}{2}B = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \tag{31}$$

and

$$\tan \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \tag{32}$$

Formula (30) may be written

$$\tan \tfrac{1}{2}A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{33}$$

If we let

$$r^* = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

we may write

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}. \tag{34}$$

Similarly

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}, \tag{35}$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c}. \tag{36}$$

When calculating the angles of a triangle, the tangents of the half angles should be used, since the complete calculation of A, B, C may be performed by taking from the tables only the four logarithms: $\log s$, $\log (s-a)$, $\log (s-b)$, and $\log (s-c)$.

7-10. Given three sides. When the three sides of a triangle are given, its solution may be effected by means of the half-angle formulas and the results checked by means of the relation $A + B + C = 180°$.

* r is the radius of the circle inscribed in the triangle.

Example. Given $a = 6.823$, $b = 5.206$, and $c = 3.163$. Find A, B, and C.

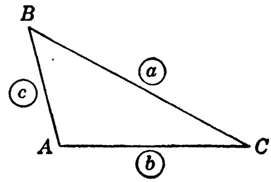


FIG. 7-26.

Solution. The half-angle formulas are

$$\tan\frac{A}{2} = \frac{r}{s-a}, \tag{a}$$

$$\tan\frac{B}{2} = \frac{r}{s-b}, \tag{b}$$

$$\tan\frac{C}{2} = \frac{r}{s-c}, \tag{c}$$

where

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{d}$$

The solution is compactly arranged in the following form:

	(d)	(a)	(b)	(c)
$a = 6.823$				
$b = 5.206$				
$c = 3.163$				
$2s = 15.192$				
$s = 7.596$	colog $s = 9.1194-10$			
$s-a = 0.773$	log $(s-a) = 9.8882-10$	colog $(s-a) = 0.1118$		
$s-b = 2.390$	log $(s-b) = 0.3784$		colog $(s-b) = 9.6216-10$	
$s-c = 4.433$	log $(s-c) = 0.6467$			colog $(s-c) = 9.3533-10$
$s = 7.596$				
	log $r^2 = 0.0327$			
	log $r = 0.0164$	log $r = 0.0164$		
$A = 106°40'$, $\frac{1}{2}A = 53°20'$		l tan $\frac{1}{2}A = 0.1284$		
			log $r = 0.0164$	
$B = 46°58'$, $\frac{1}{2}B = 23°29'$			l tan $\frac{1}{2}B = 9.6380-10$	
				log $r = 0.0164$
$C = 26°22'$, $\frac{1}{2}C = 13°11'$				l tan $\frac{1}{2}C = 9.3697-10$
$A+B+C = 180°0'$.				

The arithmetic involved in computing $s - a$, $s - b$, and $s - c$ was checked by verifying that their sum was s.

By means of the law of cosines, we can find by the use of the slide rule one of the angles of the triangle. Then, by applying the law of sines, we read on the slide rule the other two angles.

EXERCISES 7-6

Solve the following triangles:

1. $a = 3.41$, $b = 2.60$, $c = 1.58$.

2. $a = 95.32$, $b = 113.7$, $c = 179.8$.

3. $a = 111$, $b = 145$, $c = 40$.

4. $a = 14.49$, $b = 55.44$, $c = 66.91$.

5. $a = 97.86$, $b = 105.9$, $c = 138.7$.

6. $a = 2.236$, $b = 2.449$, $c = 2.646$.

7. $a = 1.493$, $b = 2.871$, $c = 1.901$.

8. $a = 529.4$, $b = 716.5$, $c = 635.2$.

9. Find the largest angle of the triangle whose sides are 13, 14, 16.

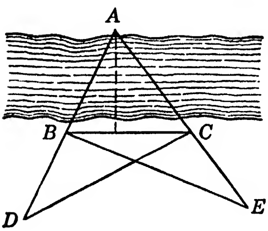


FIG. 7-27.

10. To find the width of a river, a point A is located on one bank and two points B and C on the other. A fourth point D is located in line with AB, and a fifth point E in line with AC. The distances were measured as follows: $BC = 506$ ft., $BD = 453$ ft., $BE = 809$ ft., $CD = 753$ ft., $CE = 392$ ft. Find the width of the river.

11. Three towns, A, B, and C, are situated so that $AB = 23.37$ miles, $BC = 11.84$ miles, and $AC = 16.29$ miles. A road from A to B is met at D by a perpendicular road from C. Find the length of this latter road and the distance DB.

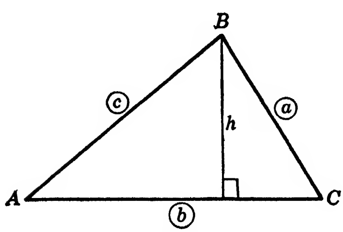


FIG. 7-28.

12. Derive Heron's formula for the area K of a triangle in terms of its three sides a, b, c, and

$$s = \tfrac{1}{2}(a + b + c),$$

namely:

$$K = \sqrt{s(s - a)(s - b)(s - c)}.$$

Hint. The area of the triangle shown in Fig. 7-28 is

$$K = \tfrac{1}{2}bh = \tfrac{1}{2}cb \sin A.$$

Replace $\sin A$ by $2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$, and then use (5) and (9).

13. Use Heron's formula to find the area of the triangle of (*a*) Exercise 1; (*b*) Exercise 7.

14. The sides of a triangular field measure 223.6 ft., 244.9 ft., and 264.6 ft. Find the area of the field.

7-11. Summary. A summary of the four cases of oblique triangles is given below in tabular form.

Given	One side and two angles	Two sides and the angle opposite one of them	Two sides and the included angle	Three sides
Using logarithms, solve by	Law of sines	Law of sines	Law of tangents and law of sines	Tangent of half-angle formulas
Using slide rule, solve by	Law of sines	Law of sines	Dropping a perpendicular	Law of cosines and law of sines
Check by	Mollweide's equations			$A + B + C = 180°$, and slide rule

7-12. Finding the area of a triangle.

a. **Given two sides and the included angle.** In Fig. 7-29, h is the altitude on side b. Sides a and b and angle C are known. The area of the triangle is $A = \frac{1}{2}hb$. Since $h/a = \sin C$, then $h = a \sin C$. Substituting for h in $\frac{1}{2}hb$, we get $A = \frac{1}{2}(a \sin C)\ b = \frac{1}{2}ab \sin C$. Expressed in words, **the area of a triangle is equal to one-half the product of the two given sides and the sine of the included angle.**

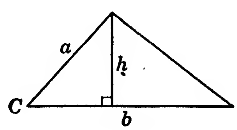


FIG. 7-29.

b. **Given two angles and a side** or **two sides and an angle opposite one of them.** By means of the law of sines, find the value of an unknown side or angle, and then apply the formula developed in the preceding paragraph.

c. **Given three sides.** From the study of plane geometry we know that the area of a triangle in terms of the sides is given in the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, in which s represents one-half the perimeter; that is, $s = \frac{1}{2}(a + b + c)$.

Example 1. Find the area of a triangle in which two sides are 12 and 10 and the included angle is 50°.

Solution. $A = \frac{1}{2}ab \sin C$
$= \frac{1}{2}(12)(10)(\sin 50°) = 60(.7660)$
$=$ **45.96,** or **46.**

Example 2. Find the area of a triangle the sides of which are 14, 16, and 12.

Solution. $2s = a + b + c = 14 + 16 + 12 = 42,$
$s = 21,$
$s - a = 7, s - b = 5, s - c = 9,$
$A = \sqrt{21 \times 7 \times 5 \times 9} =$ **81.27** or **81.**

EXERCISES 7-7

1. Find the area of each of the following triangles, given

(*a*) $a = 15, b = 20, C = 30°.$
(*b*) $c = 14, b = 18, A = 60°.$
(*c*) $a = 9.5, c = 8.4, B = 45°.$
(*d*) $b = 12.3, a = 6.9, C = 35°26'.$
(*e*) sides 12, 16, 15.
(*f*) sides 20.1, 28.6, 24.2.
(*g*) sides 42.5, 38.5, 36.4.
(*h*) sides 5.44, 8.15, 6.31.
(*i*) $a = 13.1, A = 28°15', B = 59°37'.$
(*j*) $b = 12.52, B = 51°18', C = 42°39'.$

2. Find the area of an isosceles triangle each of whose equal sides is 19.5 and the included angle is 102°.

3. Find the area of an isosceles triangle whose base is 14.6 and whose vertex angle is 48°26′.

4. Find the area of a parallelogram whose sides, 18 and 16, include an angle of 35°.

5. Two streets meet at an angle of 65°64′. How much land is there in the triangular corner lot which has a frontage of 275.3 ft. on one street and 319.8 ft. on the other?

6. Two streets meet at an angle of 25°14′. From point A on one of the streets, 1000 ft. from the intersection, a straight fence is run to the other street so that the lot so made will contain 2 acres. What will be the frontage on the second street? (One acre contains 43,560 sq. ft.)

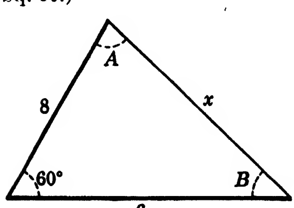


FIG. 7-30.

MISCELLANEOUS EXERCISES 7-8

1. Use the law of cosines to find x in Fig. 7-30; then express $\sin A$ and $\sin B$ in terms of x.

2. In Fig. 7-31 find $\tan \frac{1}{2}(A - B)$ by using formula (9) in Art. 7-6.

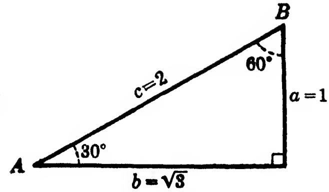


FIG. 7-31.

3. In each of these figures use the law of cosines to find x. Then express $\sin A$ and $\sin B$ in terms of x.

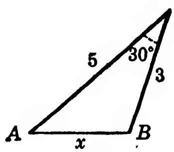


FIG 7-32.

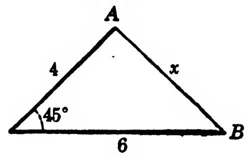


FIG. 7-33.

4. In each of these figures find $\tan \frac{1}{2}(A - B)$ by using formula (9) in Art. 7-6.

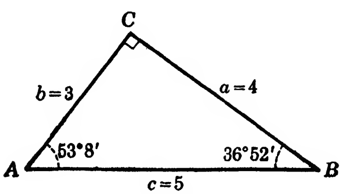


FIG. 7-34.

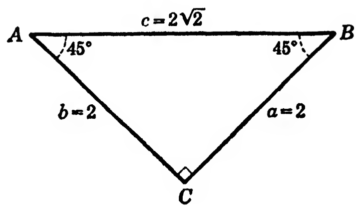


FIG. 7-35.

5. Use the law of cosines to find the value of x in Fig. 7-36.

6. Find the value of $\tan \frac{1}{2}(A - B)$ where A and B are defined by Fig. 7-36.

7. Find the area of the triangle in Fig. 7-36.

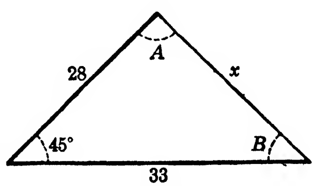


FIG. 7-36.

8. Write equations applying to Fig. 7-37 by using each of the following: law of sines, law of cosines, law of tangents, Mollweide's equations.

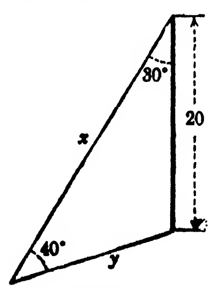


FIG. 7-37.

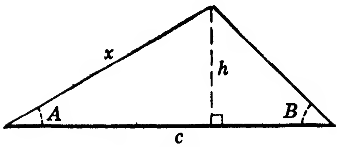


FIG. 7-38

9. Find an expression for the area of the triangle in Fig. 7-38 in terms of c, A, and B.

Hint. First find x and then h.

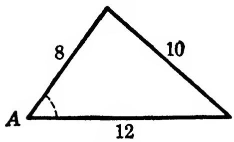


FIG. 7-39.

10. Find the value of $\cos A$ where A is defined by Fig. 7-39.

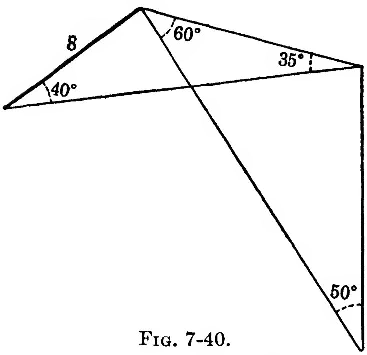


FIG. 7-40.

11. (*a*) From Fig. 7-40 find a formula for h in terms of the given parts.

(*b*) Using the formula found in (*a*), compute h.

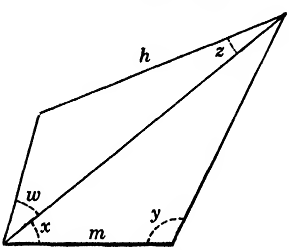


FIG. 7-41.

12. Using Fig. 7-41, express h in terms of m, x, y, z, w.

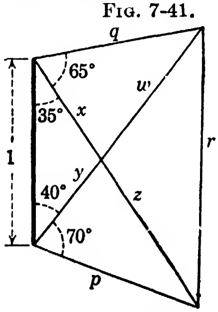


FIG. 7-42.

13. Find the length of *all* line segments of Fig. 7-42 in terms of the given parts.

14. Draw the altitude to the side lettered x in Fig. 7-43 and find its length in terms of θ and φ; then write a formula for the area of the triangle. Check this formula by using the values $\theta = 90°$, $\varphi = 45°$.

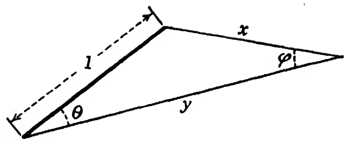


FIG. 7-43.

15. In Fig. 7-44 trihedral angle O has the face angles a, b, c, and trihedral angle C has the face angles C, 90°, 90°. Express the length of each line segment in terms of a, b, c, then find and equate two line values of DE, and simplify to obtain $\cos c = \cos a \cos b + \sin a \sin b \cos C$.

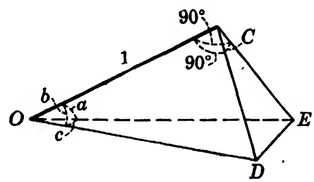


FIG. 7-44.

16. From the law of cosines derive algebraically the law of sines. *Hint.* Find $\cos A$ in terms of a, b, and c; then find

$$\frac{(\sin^2 A)}{a^2} = \frac{(1 - \cos^2 A)}{a^2}.$$

17. O-ABC in Fig. 7-45 represents a pyramid. Find the length of each edge in terms of α, β, γ, θ, and φ.

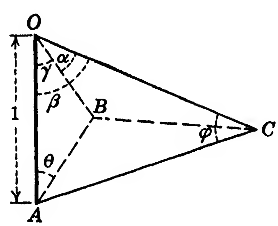


FIG. 7-45.

18. Two points A and B are inaccessible from C. If $AB = 1308$ ft., angle $CAB = 53°7'$, and angle $CBA = 70°15'$, find the distance from C to each of the other two points.

19. The angles of elevation of a balloon, directly above a straight road, from two points of the road on opposite sides of the balloon, are 78°15′ and 59°48′. If the two points are 5000 ft. apart, what is the height of the balloon?

20. A 52-ft. ladder is set against an inclined buttress and reaches 46 ft. up its face. If the foot of the ladder is 20 ft. from the foot of the inclined face, what is the inclination of the face of the buttress?

21. A and B are separated by an obstruction, but C is accessible from both. If $AC = 161.3$ ft., $CB = 793.6$ ft., and angle $C = 58°22.5'$, what is the distance AB?

22. A ship sails 23 miles on compass course 15°, thence 15 miles on compass course 78°. How far and in what direction is she from her starting point?

23. The area of a triangle whose angles are 61°9′, 34°14′ and 84°35′ is 680.60. What is the length of the longest side?

24. The captain of a ship traveling at 14 knots on compass course 66° sights a lighthouse bearing 39°. After 10 min. the lighthouse bears 17°30′. How long does it take to get to the point nearest the lighthouse, and how far away is it at that time?

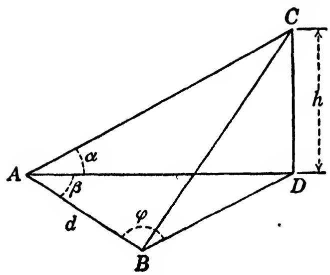


FIG. 7-46.

25. The magnitude h of an inaccessible vertical height DC is desired. A base line AB of length d in the horizontal plane through the base D of the object is laid off, and the angles DAC, DAB, and DBA are found by measurement to be α, β, and φ, respectively.

(a) Show that

$$h = d \sin \varphi \tan \alpha \csc (\beta + \varphi).$$

(b) If $d = 132.1$ ft., $\alpha = 32°16′$, $\beta = 22°35′$, $\varphi = 20°48′$, find h.

26. From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 ft. high at the foot of the hill are observed to be 45°13′ and 47°12′, respectively. Find the height of the hill.

27. The angle of elevation of a balloon ascending uniformly and vertically at a height of 1 mile is observed to be 35°20′; 20 min. later the elevation is observed to be 55°40′. How fast is the balloon moving?

28. A flagpole 160.43 ft. high is situated at the top of a hill. At a point 600 ft. down the hill the angle between the surface of the hill and a line to the top of the flagpole is 8°. Find the distance from the point to the top of the flagpole and the inclination of the ground to a horizontal plane.

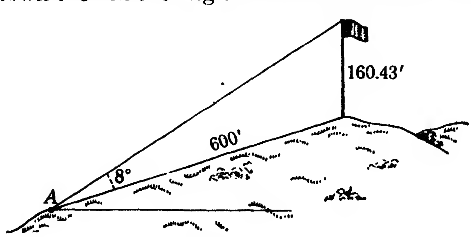


FIG. 7-47.

29. From a point on a horizontal plane the angle of elevation of the top of a mountain peak is 40°28′, and 4163 ft. farther away in the same

vertical plane the angle of elevation is 28°50′. Find the height of the peak above the horizontal plane.

30. A tower (Fig. 7-48) stands on a hill inclined 22° with the horizontal. At a point A some distance down the hill the angle of elevation

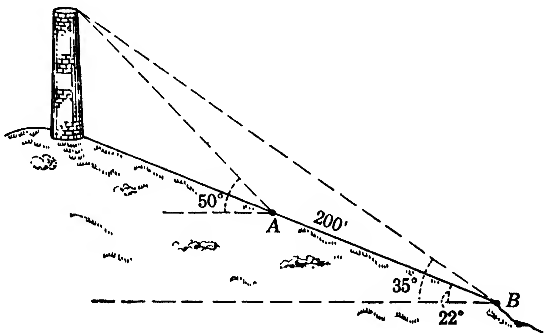


Fig. 7-48.

of the top of the tower is 50° and at B, 200 ft. farther down the hill, the angle is 35°. Find the height of the tower.

31. A tower stands at the foot of a hill inclined 18° with the horizontal. At a point A some distance up the hill the angle of elevation of the top of the tower is 28°, and at B, 120 ft. farther up the hill, the angle is 15°. Find the height of the tower.

32. From a ship two lighthouses bear N. 45°E. After the ship sails at 11 knots on a course of 130° for 2 hr., the lighthouses bear 6° and 356°, respectively. Find the distance between the lighthouses.

33. A 50-ft. vertical pole casts a shadow 62 ft. 3 in. in length along the ground when the sun's altitude is 41°38′. Find the inclination of the ground in the line of the shadow.

34. The diagonals of a parallelogram are 376.1 ft. and 427.2 ft., and the included angle is 70°12′. Find the length of the sides.

35. If R is the radius of a circle circumscribed about the triangle ABC, show that

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hint. Angle BAC = angle DOC.

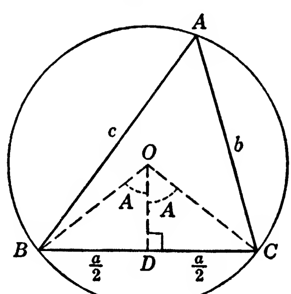


Fig. 7-49.

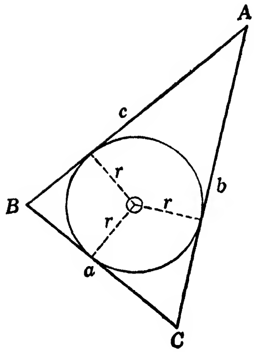


FIG. 7-50.

36. Find the radius of a circle inscribed in a triangle whose sides are a, b, and c.

Hint. The area K of the triangle ABC is $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = rs$.

37. Prove that the area K of a triangle is given by the formula

$$K = \frac{abc}{4R},$$

where R is the radius of the circumscribing circle.

38. Show that in any triangle

(a) $a^2 + b^2 + c^2 = 2(ab \cos C + bc \cos A + ac \cos B)$.

(b) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2\,abc}$.

39. An observer whose eye is 37 ft. above the surface of the water measures the compass bearing and depression of two buoys as follows: A, compass bearing 103°, depression 3°50′; B, compass bearing 165°, depression 2°45′. Find the length AB and the compass bearing of B from A.

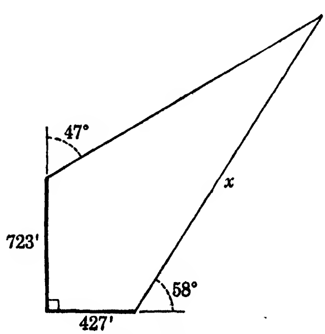


FIG. 7-51.

40. Find the value of x in Fig. 7-51.

41. Two stations, B and C, are situated on a horizontal plane 1200 ft. apart. A balloon is directly above a point A in the same horizontal plane as B and C. At B the angle of elevation of the balloon is $61°30'$, and the angle at B subtended by AC is $53°12'$, and at C the angle subtended by AB is $71°37'$. Find the height of the balloon.

42. A plane through a vertical flagpole on a small hill contains two points A and B lying 130 ft. apart in a horizontal plane, both on the same side of the hill. From A the angles of elevation of the top and bottom of the flagpole are 13° and 6°, respectively, and from B the angle of elevation of its top is 10°. Find the height of the flagpole.

43. A, B, C are three objects at known distances apart; namely, $AB = 1056$ yd., $AC = 924$ yd., $BC = 1716$ yd. An observer places himself at a station P, from which C appears directly in front of A and observes the angle CPB to be $14°24'$. Find the distance CP.

44. The foremast on a freighter sailing west bears N. 35° W. for an observer on a submarine 10,000 yd. from the mast. A torpedo fired from the submarine in a direction N. 53° W. travels at the rate of 27 knots and crosses the path of the freighter 235 yd. ahead of its mast. Find the speed of the freighter (see Fig. 7-52). (Take 2000 yd. = 1 nautical mile.)

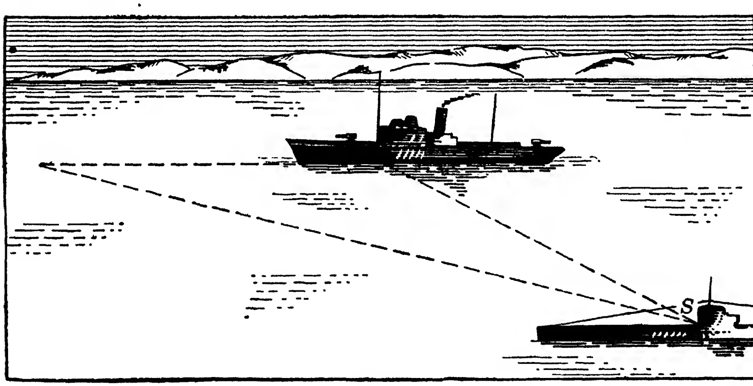


Fig. 7-52.

45. A vertical plane through the foremast of an anchored freighter cuts a hill on the near-by shore in a line AB inclined 37° to the horizontal. From A the angle of depression of the top T of the mast is 9°, and from B, 98 ft. downhill from A, the angle of elevation of T is 7°. If the mast subtends an angle of 14° at B, find its height.

46. P and Q are two inaccessible objects. A straight line AB, in the same plane with P and Q, is measured and found to be 280 yd. long. If angle $PAB = 95°$, angle $QAB = 47°30'$, angle $QBA = 110°$, and angle $PBA = 52°20'$, find the length of PQ.

47. A and B are two stations 1 mile apart, and B is due east of A. When an airplane is due north of A its angles of elevation at A and B are 37° and 23°, respectively, and when due north of B, its angles of elevation at A and B are 12° and 19°, respectively. Find its altitude at each time of observation and the compass course it is traveling.

48. On the bank of a river there is a column 200 ft. high supporting a statue 30 ft. high. The statue to an observer on the opposite bank subtends the same angle that a man 6 ft. high subtends standing at the base of the column. Find the breadth of the river.

49. From a certain station the angular elevation of a mountain peak in the northeast is observed to be α. A hill $22\frac{1}{2}°$ south of east whose height above the station is known to be h is then ascended, and the mountain peak is now seen in the north at an elevation β. Prove that the height of its summit above the first station is $h \sin \alpha \cos \beta \csc (\alpha - \beta)$.

50. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Show that, if the distance of the observer from the foot of the hill be c, the height of the tower is $\dfrac{bc \sin \alpha}{a + b + c \cos \alpha}$.

51. A body is acted upon by two forces of 5 and 3 lb. at an angle of 60°. Find the magnitude and the direction of the resultant.

52. An airplane is flying with a speed of 180 knots on a heading of 300° and a wind of 20 knots is blowing from 50°. Find the actual direction taken by the plane over the ground and the distance actually covered in 1 hr.

53. Two forces of 15 and 18 lb. are acting upon a point P. Find their resultant when the angle between them is (a) 120°, (b) 150°.

54. The angular elevation of a column as viewed from a station due north of it is α, and as viewed from a station due east of the former station and at a distance c from it is β. Prove that the height of the column is

$$\frac{c \sin \alpha \sin \beta}{[\sin (\alpha - \beta) \sin (\alpha + \beta)]^{\frac{1}{2}}}.$$

55. An observer found the angle of elevation of the summits of two spires which appears in a straight line to be α, and the angles of depression of their reflections in still water to be β and γ. If the height of the observer's eye above the level of the water was c, show that the horizontal distance between the spires is

$$\frac{2c \cos^2 \alpha \sin (\beta - \gamma)}{\sin (\beta - \alpha) \sin (\gamma - \alpha)}.$$

56. A, B, C are three objects so situated that $AB = 320$ yd., $AC = 600$ yd., and $BC = 435$ yd. From a station P it is observed that $APC = 15°$, and $BPC = 30°$. Find the distances of P from A, B, and C if the point A is nearest P and the angle APB is the sum of the angles APC and BPC.

Hint. From Fig. 7-53, $PC = 600 \sin x/\sin 15° = 435 \sin y/\sin 30°$. Solve this equation for $\sin x/\sin y$, apply composition and division, and in the result replace $\sin x - \sin y$ by $2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$ and $\sin x + \sin y$ by $2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$, and simplify to obtain

$$\tan \tfrac{1}{2}(x - y) = \frac{435 \sin 15° - 600 \sin 30°}{435 \sin 15° + 600 \sin 30°} \tan \tfrac{1}{2}(x + y). \qquad (A)$$

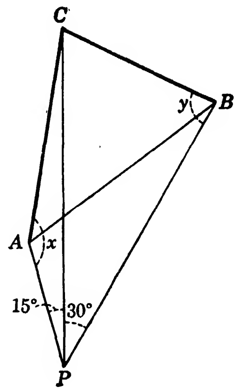


FIG. 7-53.

Compute angle C, replace $x + y$ in (A) by $360° - (15° + 30° + C)$, and solve the result for $x - y$, etc.

57. A certain gun with a shooting range of 1000 yd. per degree of elevation is pointed 20° above a horizontal plane. If a direct hit is registered on a target at a range of 20,000 yd. when the trunion axis is horizontal, find the variation in range and the variation in deflection to be expected on the second shot if for it the trunion axis is tilted through 5°.

58. Find the answer to the problem resulting when, in Exercise 57, the angle of elevation is replaced by θ, the range by R, and the angle of trunion tilt by ϕ.

59. An airplane when leaving its base flies 80 miles on course 70°12′ and then changes course to 180°. After it has traveled 27 miles on this course, find the bearing of its base and the distance to it.

60. A transport 68.2 miles due south of a lighthouse steams on course 46°58′ a distance of 31.6 miles. Find the distance and the bearing of the lighthouse from the final position.

61. A submarine is to run from point A to point B, 20 miles northeast of A. It first goes to a station C distant 15 miles and bearing 110° from north as viewed from A and then goes to B. Find the course and the distance for the second part of the trip.

CHAPTER 8

INVERSE TRIGONOMETRIC FUNCTIONS

8-1. Inverse trigonometric functions. To any angle there corresponds one and only one value of each trigonometric function, but to any value of a trigonometric function there correspond many angles. Thus $\sin 30° = \frac{1}{2}$, but 30°, 150°, 390°, and many other angles have a sine whose value is $\frac{1}{2}$.

The problem of finding the value of a trigonometric function of a given angle has already been considered in detail. The inverse problem, namely that of expressing the angles when the value of a trigonometric function is known, is the problem of this chapter. Consider the equation

$$y = \sin x. \tag{1}$$

Evidently x in this equation is an angle whose sine is y. To express this, we introduce the symbol $\sin^{-1}$,* write

$$x = \sin^{-1} y, \tag{2}$$

and read the symbol $\sin^{-1} y$ as *the angle whose sine is y*. Since the problem of finding x in equation (1) when y is given is the inverse of finding y when x is given, the symbol $\sin^{-1} y$ is often read as the *inverse sine of y* or the *arc sine of y*.

Similarly, the symbol $\cos^{-1} x$ means the angle whose cosine is x and is read the *angle whose cosine is x*, the *inverse cosine of x*, or the *arc cosine of x*. The symbols $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ are defined and read in an analogous manner.

Example. Find two positive angles x less than 360° for which (*a*) $x = \tan^{-1} 1$, (*b*) $x = \cos^{-1} (-\frac{1}{2})$.

Solution. Since the tangent of a first-quadrant angle or of a third-quadrant angle is positive, it appears that $x =$ **45°** and

* In the notation $\sin^{-1} x$, -1 is not an algebraic exponent, and $\sin^{-1} x$ does not denote $1/\sin x$. To avoid confusion, when $1/\sin x$ is meant, write $(\sin x)^{-1}$.

$x = \mathbf{225°}$ satisfy $x = \tan^{-1} 1$. The cosine of a second-quadrant angle or of a third-quadrant angle is negative; hence $x = \mathbf{120°}$ and $x = \mathbf{240°}$ satisfy $x = \cos^{-1}(-\frac{1}{2})$.

EXERCISES 8-1

For each of the following equations find two positive values of y less than 360° satisfying it:

1. $y = \sin^{-1} \frac{1}{2}$.

2. $y = \sin^{-1} \frac{1}{2}\sqrt{3}$.

3. $y = \sin^{-1}(-\frac{1}{2}\sqrt{2})$.

4. $y = \tan^{-1}\sqrt{3}$.

5. $y = \tan^{-1}(-1)$.

6. $y = \cos^{-1}(-\frac{1}{2})$.

7. $y = \cos^{-1}(-\frac{1}{2}\sqrt{2})$.

8. $y = \sec^{-1}\sqrt{2}$.

9. $y = \sec^{-1} 2$.

10. $y = \csc^{-1}(-2)$.

11. $y = \csc^{-1}\frac{2}{3}\sqrt{3}$.

12. $y = \sin^{-1} 0.4321$.

8-2. Graphs of the inverse trigonometric functions. Since

$$x = \sin y \qquad \text{and} \qquad y = \sin^{-1} x$$

express the same relation between x and y, we may make a table showing corresponding values of x and y for plotting $y = \sin^{-1} x$

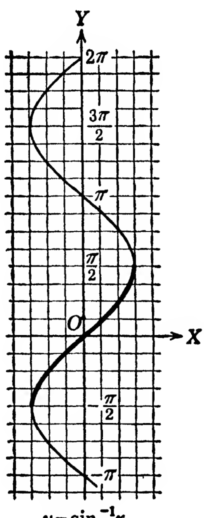


FIG. 8-1.

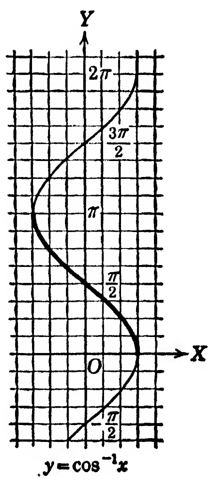


FIG. 8-2.

by using $x = \sin y$. Since this latter equation is the result of interchanging x and y in $y = \sin x$, we can obtain a table of values

for plotting $y = \sin^{-1} x$ by interchanging x and y in the table of values used in Art. 5-5 to plot $y = \sin x$. Hence, interchanging x and y in the table of Art. 5-5, plotting the points represented by the pairs of values in this new table, and connecting them by a smooth curve, we obtain the graph of $y = \sin^{-1} x$ (see Fig. 8-1).

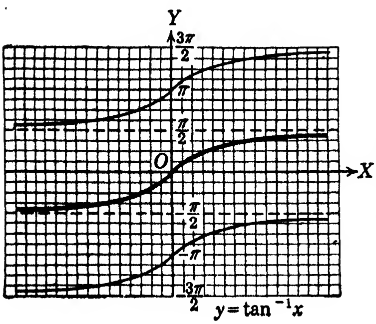


FIG. 8-3.

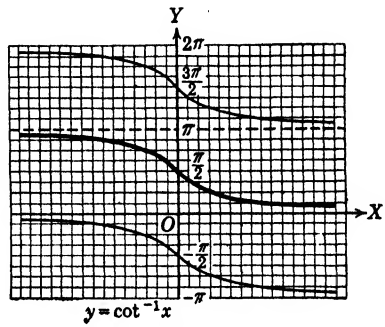


FIG. 8-4.

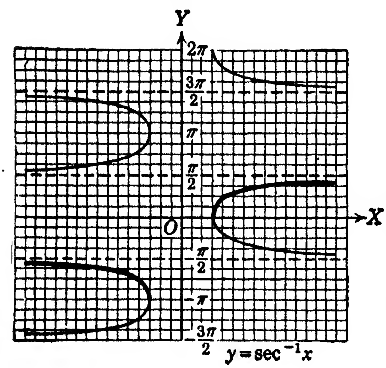


FIG. 8-5.

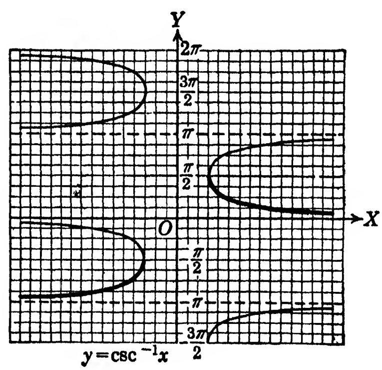


FIG. 8-6.

By a similar procedure, tables of values are prepared for plotting the other inverse trigonometric functions; their graphs are shown in Figs. 8-2 to 8-6.

EXERCISES 8-2

Construct the graphs of the following equations:

1. $y = \sin^{-1} \dfrac{x}{2}$.

2. $y = \cos^{-1} \dfrac{x}{3}$.

3. $y = \tan^{-1} 2x$.

4. $y = \cot^{-1} \dfrac{x}{2}$.

5. $y = \sec^{-1} 2x$.
6. $y = \csc^{-1} 3x$.
7. $2y = \sin^{-1} 3x$.
8. $y = 4 \cos^{-1} 2x$.
9. $y = 2 \tan^{-1} \frac{x}{3}$.
10. $\frac{1}{3}y = 2 \cot^{-1} \frac{1}{2}x$.
11. $y = \frac{1}{2} \sec^{-1} x$.
12. $y = \frac{2}{3} \csc^{-1} \frac{3}{2}x$.

8-3. Representation of the general value of the inverse trigonometric functions. In Art. 8-1, we saw that there are generally two positive values of x less than 360° satisfying an equation of the form

$$x = fn^{-1}(a), \tag{3}$$

where fn stands for sin, cos, tan, cot, sec, or csc. If α_1 and α_2 are two such values satisfying (3), then

$$x = \alpha_1 + n360° \qquad \text{and} \qquad x = \alpha_2 + n360° \tag{4}$$

satisfy (3) if n is an integer; for the six trigonometric functions of an angle are unaffected when the angle is changed by an integral multiple of 360°. When radians are used, the solution (4) is written

$$x = \alpha_1 + 2n\pi, \qquad \text{and} \qquad x = \alpha_2 + 2n\pi. \tag{5}$$

Example. Find the general value of $\sin^{-1}(-\frac{1}{2})$.

Solution. Expressed in degrees, the two positive angles less than 360° each of which has a sine equal to $-\frac{1}{2}$, are 210° and 330°. Hence the general value of $\sin^{-1}(-\frac{1}{2})$ is

$$\mathbf{210° + n360°, 330° + n360°,}$$

or, expressed in radians,

$$\frac{7\pi}{6} + n2\pi, \frac{11\pi}{6} + n2\pi.$$

EXERCISES 8-3

1. Find the general value of the angles represented by the following symbols:

(a) $\sin^{-1} \frac{1}{2}$.
(b) $\sin^{-1} \frac{1}{2}\sqrt{3}$.
(c) $\sin^{-1} \frac{1}{2}\sqrt{2}$.
(d) $\sin^{-1}(-\frac{1}{2}\sqrt{3})$.
(e) $\sin^{-1} 0$.
(f) $\sin^{-1}(-1)$.
(g) $\sin^{-1} \frac{1}{3}$.
(h) $\sin^{-1} 0.4321$.
(i) $\sin^{-1}(-\frac{5}{12})$.
(j) $\cos^{-1} \frac{1}{2}\sqrt{2}$.
(k) $\sec^{-1}(-\sqrt{2})$.
(l) $\cos^{-1}(-\frac{1}{2}\sqrt{3})$.
(m) $\csc^{-1}(-2)$.
(n) $\tan^{-1}(-1)$.
(o) $\tan^{-1} \infty$.
(p) $\cot^{-1} 1$.
(q) $\cot^{-1} \infty$.
(r) $\cot^{-1} 0.4321$.

2. For each pair of the following equations, find all values of x that satisfy both of them:

(*a*) $x = \sin^{-1}(-\frac{1}{2})$,	$x = \cos^{-1} \frac{1}{2}\sqrt{3}$.
(*b*) $x = \tan^{-1} \frac{1}{3}\sqrt{3}$,	$x = \sin^{-1}(-\frac{1}{2})$.
(*c*) $x = \sin^{-1} \frac{1}{2}\sqrt{2}$,	$x = \tan^{-1}(-1)$.
(*d*) $x = \sec^{-1}(-\sqrt{2})$,	$x = \cot^{-1} 1$.
(*e*) $x = \csc^{-1} 2$,	$x = \cot^{-1}(-\sqrt{3})$.
(*f*) $x = \cos^{-1} \frac{1}{2}$,	$x = \csc^{-1}(-\frac{2}{3}\sqrt{3})$.

3. Find the general value of the angles represented by the following symbols:

(*a*) $\sin^{-1} 0.36$.	(*b*) $\cos^{-1} 0.60$.
(*c*) $\tan^{-1} 0.90$.	(*d*) $\cot^{-1} 2.1$.
(*e*) $\sec^{-1} 3.42$.	(*f*) $\csc^{-1} 1.21$.
(*g*) $\cos^{-1} \frac{3}{5}$.	(*h*) $\sin^{-1} \frac{2}{3}$.
(*i*) $\tan^{-1} \frac{5}{4}$.	(*j*) $\sec^{-1} \frac{3}{2}$.
(*k*) $\cot^{-1} \frac{7}{8}$.	(*l*) $\csc^{-1} 15$.

8-4. Principal values. Of the many values of an inverse trigonometric function, a special one is often called the *principal value*. Many ways of choosing a principal value could be devised. The choice dictated by advanced mathematics may be obtained by using the following statements.

Let a represent a *positive* number throughout this article. The *principal value* of $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a$, etc., (if it exists) is zero or a positive angle no greater than 90°. For example, the principal value of $\sin^{-1} \frac{1}{2}$ is 30°, that of $\cos^{-1} 1$ is zero, and that of $\tan^{-1} 1$ is 45°.

The principal value of $\sin^{-1}(-a)$ (if it exists) or of $\tan^{-1}(-a)$ is a negative angle no greater numerically than 90°. For example, the principal value of $\sin^{-1}(-\frac{1}{2})$ is $-30°$, and that of $\tan^{-1}(-1)$ is $-45°$.

The principal value of $\cos^{-1}(-a)$ (if it exists) or of $\cot^{-1}(-a)$ is either 90°, 180°, or a positive second-quadrant angle. For example, the principal value of $\cos^{-1}(-1/\sqrt{2})$ is 135°, that of $\cot^{-1}(-1)$ is 135°, and that of $\cos^{-1}(-1)$ is 180°.

The principal value (if it exists) of $\sec^{-1}(-a)$ or $\csc^{-1}(-a)$ is a negative angle lying between $-90°$ and $-180°$. For example, the principal value of $\sec^{-1}(-2)$ is $-120°$, that of $\csc^{-1}(-\sqrt{2})$ is $-135°$, and that of $\csc^{-1}(-1)$ is $-90°$.

Figure 8-7 may help in choosing principal values. In Art. 8-2, the part of each graph drawn with a heavy line is the graph repre-

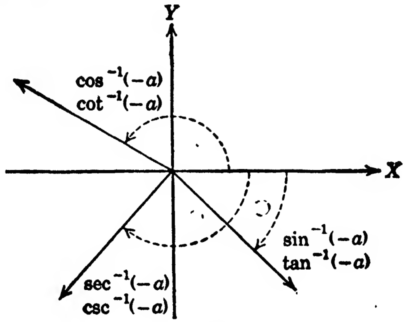


FIG. 8-7.

senting the principal value of the associated inverse trigonometric function.

EXERCISES 8-4

1. Find the principal values of the following:

(a) $\sin^{-1} \frac{1}{2}\sqrt{2}$. (b) $\sin^{-1} \frac{1}{2}\sqrt{3}$. (c) $\sin^{-1} 0$.
(d) $\tan^{-1} 1$. (e) $\tan^{-1} \sqrt{3}$. (f) $\tan^{-1} 0$.
(g) $\cot^{-1} 1$. (h) $\cos^{-1} \frac{1}{2}$. (i) $\cos^{-1} \frac{1}{2}\sqrt{2}$.
(j) $\cos^{-1} 0$. (k) $\cos^{-1} \frac{1}{2}\sqrt{3}$. (l) $\csc^{-1} \frac{2}{3}\sqrt{3}$.
(m) $\csc^{-1} 1$. (n) $\cot^{-1} \sqrt{3}$. (o) $\sec^{-1} 2$.
(p) $\cos^{-1} 1$. (q) $\sec^{-1} \frac{2}{3}\sqrt{3}$. (r) $\cot^{-1} \dfrac{1}{\sqrt{3}}$.

2. Find the principal values of the following:

(a) $\sin^{-1} (-\frac{1}{2})$. (b) $\sin^{-1} \left(-\dfrac{1}{\sqrt{2}}\right)$
(c) $\sin^{-1} \left(-\dfrac{\sqrt{3}}{2}\right)$. (d) $\tan^{-1} (-1)$.
(e) $\tan^{-1} (-\sqrt{3})$. (f) $\tan^{-1} \left(-\dfrac{1}{\sqrt{3}}\right)$.

3. Find the principal values of the following:

(a) $\cos^{-1} \left(-\dfrac{1}{\sqrt{2}}\right)$. (b) $\cos^{-1} \left(-\dfrac{\sqrt{3}}{2}\right)$.
(c) $\cos^{-1} (-\frac{1}{2})$. (d) $\cot^{-1} (-1)$.
(e) $\cot^{-1} (-\sqrt{3})$. (f) $\cot^{-1} \left(-\dfrac{1}{\sqrt{3}}\right)$.

4. Find the principal values of the following:

(a) $\sin^{-1}(-\frac{1}{2})$. (b) $\tan^{-1} 1$. (c) $\cot^{-1}(-\sqrt{3})$.
(d) $\cos^{-1} 0$. (e) $\csc^{-1}(-\sqrt{2})$. (f) $\sec^{-1}(-1)$.
(g) $\tan^{-1}(\sin 270°)$. (h) $\cot^{-1} \frac{1}{3}\sqrt{3}$. (i) $\sin^{-1} \frac{1}{2}\sqrt{3}$.
(j) $\sec^{-1} - \sqrt{2}$. (k) $\cos^{-1}(-1)$.

5. Using principal values, evaluate the following expressions, giving your answer in radian measure:

(a) $\sin^{-1}(\frac{1}{2}) - \sin^{-1}(-\frac{1}{2})$.
(b) $\sin^{-1}(-1) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.
(c) $\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.
(d) $\cos^{-1}(\frac{1}{2}) - \cos^{-1}(-\frac{1}{2})$.
(e) $\sec^{-1}(1) - \sec^{-1}(-1)$.
(f) $\csc^{-1}(-2) - \sin^{-1}(-\frac{1}{2})$.

6. Verify for principal values the following equations:

(a) $\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2}\sqrt{3} = -\sin^{-1}(-1)$.
(b) $\sin^{-1}\frac{1}{2}\sqrt{2} - 3\sin^{-1}\frac{1}{2}\sqrt{3} = -\frac{3}{4}\pi$.
(c) $\sin^{-1}(-\frac{1}{2}) + \sin^{-1}\frac{1}{2}\sqrt{2} = \frac{1}{12}\pi$.
(d) $\sin^{-1}\frac{1}{2}\sqrt{2} - \sin^{-1}\frac{1}{2}\sqrt{3} = \sin^{-1}\frac{1}{2} - \frac{1}{4}\pi$.
(e) $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2} = \sin^{-1} 1$.
(f) $\tan^{-1} 1 + \tan^{-1}\frac{1}{3}\sqrt{3} = \frac{9}{12}\pi - \tan^{-1}\sqrt{3}$.
(g) $\tan^{-1}\infty - \sin^{-1}\frac{1}{2}\sqrt{2} = \tan^{-1}\sqrt{3} - \frac{1}{12}\pi$.
(h) $\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{2} = \tan^{-1} 1 + \cos^{-1}\frac{1}{2}\sqrt{2}$.
(i) $\sin^{-1}\frac{1}{2} - \cos^{-1}(-\frac{1}{2}) = \cot^{-1}\sqrt{3} + \sec^{-1}(-2)$.

8-5. Examples involving inverse trigonometric functions. The solutions of many trigonometric equations are effected by employing the relations existing among the inverse trigonometric functions. When solving an equation involving inverse functions, the student will find it advantageous to draw a right triangle for each of the angles involved in the original equation, and designate the lengths of the sides appropriately. From these triangles the value of any desired trigonometric function is taken directly. The following examples will illustrate the method.

Example 1. Find the value of $\cos(\sin^{-1}\frac{3}{5})$ using the principal value of $\sin^{-1}\frac{3}{5}$.

Solution. Let α represent the principal value of $\sin^{-1}\frac{3}{5}$. The right triangle exhibiting α is shown in Fig. 8-8 with the sides

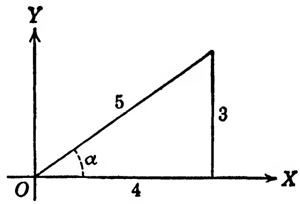


FIG. 8-8.

appropriately numbered. From this figure we read directly

$$\cos(\sin^{-1}\tfrac{3}{5}) = \cos\alpha = \frac{4}{5}.$$

Example 2. Using principal values for the inverse functions involved, find

$$\cos[\cos^{-1}(-\tfrac{1}{3}) + \sin^{-1}(-\tfrac{1}{4})]. \qquad (a)$$

Solution. Let α represent the principal value of $\cos^{-1}(-\frac{1}{3})$ and β the principal value of $\sin^{-1}(-\frac{1}{4})$. Substitution of these

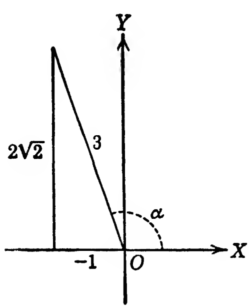


FIG. 8-9*a*.

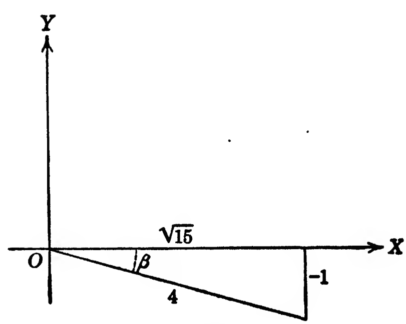


FIG. 8-9*b*.

values in (a) gives $\cos(\alpha+\beta)$. Expanding this, we obtain

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad (b)$$

Consider the two right triangles in Fig. 8-9, one exhibiting angle α, the other angle β. In accordance with the definitions

of principal values we must take α in the second quadrant and β in the fourth quadrant.

Reading the values of $\cos\alpha$, $\cos\beta$, etc., direct from the triangles and substituting them in (*b*), we obtain

$$\left(-\frac{1}{3}\right)\left(\frac{\sqrt{15}}{4}\right)-\left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{4}\right)=\frac{-\sqrt{15}+2\sqrt{2}}{12}.$$

Example 3. Show that

$$\tan^{-1}\left(-\frac{2}{9}\right)+\sin^{-1}\left(-\frac{1}{\sqrt{17}}\right)=\tfrac{1}{2}\cos^{-1}(-0.6)-90°, \qquad (a)$$

provided principal values for the inverse functions are used.

Solution. Let $A=\tan^{-1}(-\frac{2}{9})$, $B=\sin^{-1}(-1/\sqrt{17})$, $C=\cos^{-1}(-0.6)$. From these and the conventions of Art.

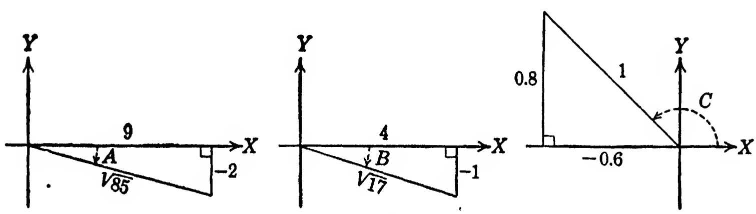


FIG. 8-10.

8-3, it appears that angles A, B, and C are correctly represented in Fig. 8-10. Inspection shows that the two members of equation (*a*) are negative acute angles. Hence they are equal if a trigonometric function of one member is equal to the same trigonometric function of the other. Equation (*a*) may be written

$$A+B=\tfrac{1}{2}C-90°. \qquad (b)$$

The cosine of the left-hand member of (*b*) is

$$\cos(A+B)=\cos A\cos B-\sin A\sin B, \qquad (c)$$

and the cosine of the right-hand member of (*b*) is

$$\cos(\tfrac{1}{2}C-90°)=\sin\tfrac{1}{2}C=\sqrt{\tfrac{1}{2}(1-\cos C)}. \qquad (d)$$

Replacing the functions in (*c*) and (*d*) by their values read from Fig. 8-10, we have

$$\cos (A + B) = \left(\frac{9}{\sqrt{85}}\right)\left(\frac{4}{\sqrt{17}}\right) - \left(\frac{-2}{\sqrt{85}}\right)\left(\frac{-2}{\sqrt{17}}\right)$$

$$= \frac{34}{17\sqrt{5}} = \frac{2}{\sqrt{5}},$$

$$\cos (\tfrac{1}{2}C - 90°) = \sqrt{\frac{1 + 0.6}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

Since these values are equal, equation (*a*) is true.

EXERCISES 8-5

Using principal values for the inverse functions involved, evaluate the following expressions:

1. $\sin (\sin^{-1} \frac{2}{3})$.
2. $\cos (\cos^{-1} \frac{3}{5})$.
3. $\sin (\cos^{-1} \frac{5}{12})$.
4. $\cos (\sin^{-1} \frac{2}{3})$.
5. $\csc [\tan^{-1} (-\sqrt{7})]$.
6. $\sin [\sec^{-1} (-\frac{5}{3})]$.
7. $\cos [\csc^{-1} (-\frac{5}{4})]$.
8. $\cos [\cot^{-1} (-\frac{3}{4})]$.
9. $\cos [\tan^{-1} (-\frac{1}{2})]$.
10. $\sec (\cot^{-1} 2)$.
11. $\tan [\cot^{-1} (\pm 1)]$.
12. $\sec [\cot^{-1} (5.4)]$.
13. $\cos (2 \tan^{-1} 1)$.
14. $\tan (\cos^{-1} \frac{3}{5})$.
15. $\sin (\cot^{-1} \frac{1}{4})$.

16. Evaluate the following expressions, using principal values:

(*a*) $\tan [\tan^{-1} \frac{1}{2} + \tan^{-1} (-\frac{2}{3})]$.
(*b*) $\sec (\cos^{-1} \frac{1}{2} - \sin^{-1} \frac{1}{2})$.
(*c*) $\csc [\sin^{-1} (1/\sqrt{2}) + \tan^{-1} 1]$.
(*d*) $\sin [\sec^{-1} (-2) - \sin^{-1} (-\frac{3}{5})]$.

Using principal values for the inverse functions involved, verify the following equations:

17. $\sin^{-1} 1 - \tan^{-1} = \dfrac{\pi}{4}$.

18. $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{1}{4}\pi$. (Clausen's formula for finding the value of π.)

19. $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{1}{4}\pi$. (Machin's formula for finding the value of π.)

20. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x + y}{1 - xy}$.

8-6. Trigonometric equations. An equation that involves one or more trigonometric functions of a variable angle is a

trigonometric equation. A trigonometric identity is a trigonometric equation that holds true for all values of the variable for which the members of the equation are defined. On the other hand, a trigonometric equation that is satisfied by only particular values of the variable is a trigonometric equation of condition. The problem connected with an identity concerns the proof that it is invariably true, whereas the problem associated with an equation of condition is to discover for what values it is true. By a solution of a trigonometric equation we mean general expressions defining all values of the variable that will satisfy the given equation. This will mean in many problems that a number n representing any integer must be used.

There are a number of methods for solving trigonometric equations. It is often possible to express all trigonometric functions involved in terms of a single function, solve the resulting equations for this function, and then write the angles associated with the values of the function. Another method consists in transferring all terms of the given equation to the left-hand member, factoring the resulting left-hand member, equating the factors to zero, and solving each equation thus obtained. The following examples will illustrate these methods of procedure.

Example 1. Solve $2\cos^2 x + \sin x - 1 = 0.$

Solution. Replacing $\cos^2 x$ by $1 - \sin^2 x$ and simplifying slightly, we obtain

$$2(\sin x)^2 - (\sin x)^1 - 1 = 0.$$

Evidently this is a quadratic equation with $\sin x$ appearing as the unknown. Solving it by factoring, we obtain

$$(\sin x - 1)(2\sin x + 1) = 0.$$
$$\therefore \sin x = 1 \text{ or } -\tfrac{1}{2}.$$

If the function involved in the equation is not factorable, the equation may be solved by the formula* for solving quadratic equations. Thus,

$$\sin x = \frac{-(-1) \pm \sqrt{1+8}}{4} = 1 \text{ or } -\tfrac{1}{2}.$$

* The solution of $ay^2 + by + c = 0$ is $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence $x = \sin^{-1} 1$ and $x = \sin^{-1}(-\frac{1}{2})$. Replacing these inverse functions by their general values, we get

$$\mathbf{x = 90° + n360°,} \qquad \mathbf{x = 210° + n360°,} \qquad \mathbf{x = 330° + n360°}$$

or, in radians,

$$x = \frac{\pi}{2} + 2n\pi, \qquad x = \tfrac{7}{6}\pi + 2n\pi, \qquad x = \frac{11\pi}{6} + 2n\pi.$$

Example 2. Solve $\sin 4\theta + \cos 2\theta = 0$.

Solution. Replacing $\sin 4\theta$ by $2 \sin 2\theta \cos 2\theta$ in the given equation and factoring, we obtain

$$\cos 2\theta\,(2 \sin 2\theta + 1) = 0.$$

Equating the factors to zero, we get

$$\cos 2\theta = 0, \qquad 2 \sin 2\theta + 1 = 0.$$

From $\cos 2\theta = 0$ we derive

$$2\theta = 90° + n360° \qquad \text{and} \qquad 2\theta = 270° + n360°,$$

or

$$\boldsymbol{\theta = 45° + n180°} \qquad \text{and} \qquad \boldsymbol{\theta = 135° + n180°.}$$

From $2 \sin 2\theta + 1 = 0$, or $\sin 2\theta = -\frac{1}{2}$, we derive

$$2\theta = 210° + n360° \qquad \text{and} \qquad 2\theta = 330° + n360°,$$

or,

$$\boldsymbol{\theta = 105° + n180°} \qquad \text{and} \qquad \boldsymbol{\theta = 165° + n180°.}$$

EXERCISES 8-6

1. Find the values of x between 0° and 360° for which

(*a*) $\sin^2 x = \frac{1}{4}$.
(*b*) $\csc^2 x = 2$.
(*c*) $\tan^2 x - 3 = 0$.
(*d*) $\sec^2 x - 4 = 0$.
(*e*) $\tan 2x = 1$.
(*f*) $2 \sin 3x = 1$.

2. Find the values of the unknown between 0° and 360° for which

(*a*) $2 \sin^2 x + 3 \cos x = 0$.
(*b*) $\cos^2 \alpha - \sin^2 \alpha = \frac{1}{2}$.
(*c*) $2\sqrt{3} \cos^2 \alpha = \sin \alpha$.
(*d*) $\sin^2 y - 2 \cos y + \frac{1}{4} = 0$.
(*e*) $4 \sec^2 y - 7 \tan^2 y = 3$.
(*f*) $\tan B + \cot B = 2$.
(*g*) $\sin x + \cos x = 0$.

3. Find, in radians, all angles between 0 and 2π that satisfy the following equations:

(a) $(\tan x + 1)(\sqrt{3} \cot x - 1) = 0.$
(b) $(2 \cos x + 1)(\sin x - 1) = 0.$
(c) $(4 \cos^2 \theta - 3)(\csc \theta + 2) = 0.$
(d) $2 \cot \theta \sin \theta + \cot \theta = 0.$

4. For each of the following equations, find all values of the unknown that satisfy it:

(a) $2 \sin^2 x + \cos x - 1 = 0.$
(b) $2 \cos^2 \theta + 5 \sin \theta - 4 = 0.$
(c) $\cos^2 x + 2 \sin x + 2 = 0.$
(d) $2 \cos^2 2\alpha + \sin 2\alpha - 1 = 0.$
(e) $2 \sec^2 \theta - \tan \theta = 5.$
(f) $2 \csc^2 \phi - 5 \cot \phi + 1 = 0.$
(g) $4 \sec^2 2A = 8 + 15 \tan 2A.$
(h) $\cos^2 x(4 \cos^2 x - 1) = 0.$
(i) $4 \cos 2x + 3 \cos x = 1.$
(j) $\cot^2 \theta - 3 \csc \theta + 3 = 0.$
(k) $\tan^2 x + \cot^2 x - 2 = 0.$
(l) $\tan x + 3 \cot x = 4.$
(m) $2 \tan^2 x + 3 \sec x = 0.$
(n) $\cos \theta + 6 \sin \theta = 2.$
(o) $\sin x + \cos x = 1.$
(p) $\csc x \cot x = 2\sqrt{3}.$
(q) $\sin x \cos x + \frac{1}{4} = 0.$
(r) $\cos 2x + \cos x = -1.$
(s) $\tan 2\theta \tan \theta = 1.$

5. Solve for the unknown:

(a) $2 \sin \theta = \tan \theta.$
(b) $\sin 2x - \cos x = 0.$
(c) $4 \sin^4 \theta = 3 \sin^2 \theta.$
(d) $\sin 2\alpha + \cos \alpha = 0.$
(e) $\sin 4x = \cos 2x.$
(f) $\sin 2\theta = \sqrt{3} \sin \theta.$
(g) $\sin^2 4\alpha = \sin^2 2\alpha.$
(h) $2 \sin 4\theta + \sin 2\theta = 0.$
(i) $\cos 4\alpha = \cos 2\alpha.$

6. Find the abscissas of the points where each of the following curves crosses the x-axis:

(a) $y = 2 \sin x - \sin 2x.$
(b) $y = \cos 2x - \cos x.$
(c) $y = \cos 2x - \cos^2 x.$
(d) $y = \tan (x + 45°) - 1 + \sin 2x.$

7. Plot each of the following pairs of curves on the same set of axes and find their points of intersection for values of x between 0° and 360°.

(a) $y = \sin 2x,$ $y = \sin x.$
(b) $y = \cos 2x,$ $y = \cos x.$
(c) $y = \sec x,$ $y = 2 \cos x.$
(d) $y = \tan x,$ $y = 3 \cot x.$
(e) $y = 2 \sin x,$ $y = \tan x.$
(f) $y = \tan^2 x,$ $y = 2 - \cot^2 x.$

MISCELLANEOUS EXERCISES 8-7

1. Find the values of the following:

(*a*) $\sin (\tan^{-1} \frac{5}{12})$.
(*b*) $\sin (\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3})$.
(*c*) $\tan (2 \tan^{-1} a)$.
(*d*) $\cot (2 \text{ arc sin } \frac{3}{5})$.
(*e*) $\cos (2 \text{ arc cos } a)$.
(*f*) $\cos (2 \text{ arc tan } a)$.
(*g*) $\text{arc tan } \frac{1}{\sqrt{3}}$.
(*h*) $\cot^{-1} (\pm 1)$.

2. Prove the following, using principal values:

(*a*) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$.
(*b*) $\text{arc cos } \frac{4}{5} + \text{arc tan } \frac{3}{5} = \text{arc tan } \frac{27}{11}$.
(*c*) $2 \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{12}{5}$.
(*d*) $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} = \sin^{-1} \frac{77}{85}$.
(*e*) $\text{arc cos } \frac{4}{5} + \text{arc cos } \frac{12}{13} = \text{arc cos } \frac{33}{65}$.
(*f*) $\text{arc tan } \frac{1}{7} + \text{arc tan } \frac{1}{13} = \text{arc tan } \frac{2}{9}$.

Solve the following equations:

3. (*a*) $\sin x = 3 \cos x$.
(*b*) $2 \cos x = \cos 2x$.
(*c*) $\tan x = \tan 2x$.

4. (*a*) $3 \cos^2 x + 5 \sin x - 1 = 0$.
(*b*) $3 \sin x \tan x - 5 \sec x + 7 = 0$.
(*c*) $\tan x + \sec^2 x - 3 = 0$.
(*d*) $\sin x + \cos 2x = 4 \sin^2 x - 1$.
(*e*) $\sin (2x - 180°) = \cos x$.
(*f*) $\cos^2 x + 2 \sin x = 0$.
(*g*) $\sec^2 x - 4 \tan x = 0$.
(*h*) $\sin^2 2x - \sin 2x - 2 = 0$.
(*i*) $\tan^2 \frac{x}{2} - \tan \frac{x}{2} - 2 = 0$.
(*j*) $\sin x \sin \frac{x}{2} = 1 - \cos x$.
(*k*) $\csc y + \cot y = \sqrt{3}$.
(*l*) $6 \sec^2 \alpha + \cot^2 \alpha = 11$.

5. (*a*) $4 \sin x + 3 \cos x = 3$.
(*b*) $5 \sin x = 4 \cos x + 4$.

6. (*a*) $\sin(60° - x) - \sin(60° + x) = \frac{\sqrt{3}}{2}$.

(*b*) $\sin(30° + x) - \cos(60° + x) = -\frac{\sqrt{2}}{3}$.

(*c*) $\tan(45° - x) + \cot(45° - x) = 4$.

(*d*) $\sec(x + 120°) + \sec(x - 120°) = 2$.

(*e*) $\csc^2 x(1 + \sin x \cot x) = 2$.

7. (*a*) If $x = a\cos\varphi$, $y = b\sin\varphi$, prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Hint. Solve for $\sin\varphi$ and $\cos\varphi$ and then use $\sin^2\varphi + \cos^2\varphi = 1$.

(*b*) If $x = a\sec\varphi$, $y = a\tan\varphi$, prove that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(*c*) From $x = a\cos^3\varphi$, $y = a\sin^3\varphi$, deduce $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(*d*) If $x = a + b\cos\varphi$, $y = c + d\sin\varphi$, find a relation between x and y.

(*e*) From $x = a\tan^3\varphi$, $y = b\sec^3\varphi$, deduce a relation between x and y.

(*f*) If $a\sin\theta + b\cos\theta = h$, $a\cos\theta - b\sin\theta = k$, prove that $a^2 + b^2 = h^2 + k^2$.

8. Solve the following equations:

(*a*) $\tan^{-1} x + \tan^{-1}(1 - x) = \tan^{-1}(\frac{4}{3})$.

(*b*) $\text{arc tan } x + 2 \text{ arc cot } x = \frac{2\pi}{3}$.

(*c*) $\tan^{-1}\frac{x - 1}{x + 2} + \tan^{-1}\frac{x + 1}{x + 2} = \frac{\pi}{4}$.

(*d*) $\cos^{-1}\frac{x^2 - 1}{x^2 + 1} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$.

(*e*) $\text{arc tan}\frac{x + 1}{x - 1} + \text{arc tan}\frac{x - 1}{x} = \text{arc tan}(-7)$.

(*f*) $\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\frac{8}{31}$.

(*g*) $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$.

(*h*) $\text{arc sin}\frac{5}{x} + \text{arc sin}\frac{12}{x} = \frac{\pi}{2}$.

9. Plot each of the following pairs of curves on the same set of axes, and find their points of intersection between 0° and 360°.

(*a*) $y = \sin x$,	$y = \tan x$.
(*b*) $y = 2\sin x$,	$y = \tan 2x$.
(*c*) $y = \tan x$,	$y = 4 - 3\cot x$.
(*d*) $y = \cos 2x$,	$y = -(1 + \cos x)$.

CHAPTER 9

THE RIGHT SPHERICAL TRIANGLE

9-1. Introduction. Just as plane trigonometry has for its object the study of the relations existing among the sides and angles of a plane triangle, so spherical trigonometry has for its

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Chart your course right

object the study of the relations connecting the sides and angles of a spherical triangle. Since the earth is approximately a sphere, this theory will apply when distances and directions on the earth are in question. Hence the subject of spherical trigonometry is basic in navigation.

Since a spherical triangle is formed on a spherical surface, we shall review the facts and principles about a sphere that must serve as a background for the work in spherical trigonometry.

9-2. The sphere. A sphere is the locus of all points in space that are at a given distance from a fixed point called the **center** of the sphere. The given distance is the **radius** of the sphere. Thus, in Fig. 9-1, O is the center of the sphere and OA, OB, OC, and OP are radii. Straight line PQ is a **diameter** of the sphere.

The intersection of a plane with a sphere is a circle.* If the plane passes through the center of the sphere, the intersection is called a **great circle.** Other intersections are called **small circles.** Thus, in Fig. 9-1, circles $PAQD$ and $ACBD$ are great circles, since their center is O, the center of the sphere. Also, on the earth, considered as a sphere, the equator and the meridians are examples of great circles; the parallels of latitude are examples of small circles.

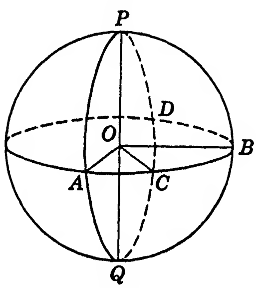


FIG. 9-1.

If a diameter of a sphere is perpendicular to the plane of a great circle, the ends of the diameter are called the **poles of the** great circle. Thus, in Fig. 9-1, P and Q are the poles of the great circle $ACBD$, if PQ is assumed to be perpendicular to the plane of $ACBD$. If the distance from a pole to its great circle is measured on another great circle passing through its pole, it is evident that a pole is a quadrant's distance from its great circle, since a quadrant is a quarter of a circle. Thus, in Fig. 9-1, arcs PA, QA, PD, and QD are **quadrants.** Each of these is an arc of 90°.

9-3. The spherical triangle. A spherical triangle consists of three arcs of great circles that form the boundaries of a portion of a spherical surface. The vertices of the spherical triangle will be denoted by capital letters A, B, and C and the sides opposite by a, b, and c, respectively. Since the sum of the angles of a spherical triangle is more than 180° and less than 540°,* the triangle may have one, two, or three right angles. A right

* These theorems are proved in solid geometry.

spherical triangle is one which has one right angle. An oblique spherical triangle has no right angles. In general, we shall consider only spherical triangles, each of whose sides and each of whose angles is less than 180°.

9-4. Important propositions from solid geometry.

1. The sum of the angles of a spherical triangle is greater than 180° and less than 540°; that is, $180° < A + B + C < 540°$.
2. If two angles of a spherical triangle are equal, the sides opposite are equal; and conversely.
3. If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle; and conversely.
4. The sum of two sides of a spherical triangle is greater than the third side.
5. The sum of the face angles of a trihedral angle is less than 360°.

9-5. The spherical triangle and its trihedral angle. In Fig. 9-2, triangle ABC is a spherical triangle formed by the arcs of three intersecting great circles, the planes of which pass through the center O of a sphere with radii OA, OB, and OC, forming the **trihedral angle** O-ABC. Since OAB is a sector of a circle with center at O, angle $AOB \stackrel{\circ}{=}$ arc AB.* Likewise, angle $AOC \stackrel{\circ}{=}$ arc AC, and angle $COB \stackrel{\circ}{=}$ arc BC. This follows from the theorem in plane geometry that an arc of a circle is measured by the angle that it subtends at the center and is expressed in degrees and minutes. Angles AOB, AOC, and COB are the **face angles** of the trihedral angle O-ABC.

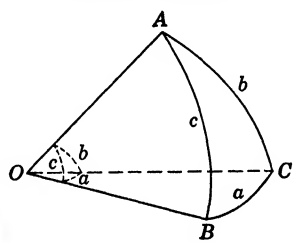


FIG. 9-2.

In Fig. 9-3, BD is tangent to arc AB at B in the plane of OBA, and BE is tangent to arc BC at B in the plane of OBC. Angle DBE by definition is the measure of the

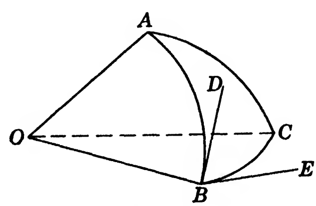


FIG. 9-3.

* The symbol $\stackrel{\circ}{=}$ is read, "contains the same number of degrees as."

angle formed by the arcs AB and BC. Likewise, **the measure of each angle of a spherical triangle is the angle formed by the two tangents drawn to the intersecting arcs at their vertex in the two respective planes.**

Since DB is perpendicular to radius OB in plane OAB, and BE is perpendicular to radius OB in plane COB, then angle DBE is the **plane angle** of the dihedral angle formed by the faces AOB and COB. Angle DBE, by definition, is the measure of the dihedral angle in which it is drawn. Hence, the dihedral angle formed by the faces AOB and BOC, or its plane angle DBE, is equal to angle B of the spherical triangle. Thus, the plane angle of each of the dihedral angles in the figure is equal to one of the angles of the spherical triangle.

The sum of the three face angles of the trihedral angle O-ABC is less than 360°. Since angle $AOB \stackrel{\circ}{=}$ arc AB, angle $AOC \stackrel{\circ}{=}$ arc AC, and angle $COB \stackrel{\circ}{=}$ arc BC, it follows that the sum of the sides of a spherical triangle is less than 360°.

EXERCISES 9-1

1. If each angle of a spherical triangle is a right angle, what is the value of each side?

2. Show that if a spherical triangle has two right angles, the sides opposite these angles are quadrants and the third angle has the same measure as the opposite side.

3. The face angles of the trihedral angle associated with a spherical triangle are each 90° and the radius of the sphere is 10 in. Find the angles of the triangle in degrees, and find the sides both in degrees and in inches.

4. Find the magnitude of the face angles and of the dihedral angles of the trihedral angle associated with a spherical triangle whose sides are 90°, 90°, and 60°.

5. The face angles of a trihedral angle at the center of the earth are 50°, 60°38′, 45°50′. Find in nautical miles* the lengths of the sides of the associated spherical triangle on the surface of the earth.

6. Two great circles on a sphere intersect at an angle of 23°30′. Find the least great-circle distance from the pole of one to a point on the other.

7. What can be said regarding the size and shape of a spherical equiangular triangle if the sum of its angles is (*a*) nearly equal to 180°? (*b*) nearly equal to 540°?

* A nautical mile is the length of an arc of a great circle on a sphere the size of the earth subtended by an angle of 1′ at its center.

8. Find all sides and angles of a spherical triangle having as angles $A = 90°$, $B = 90°$, and

(*a*) $C = 30°$. (*b*) $C = 45°$. (*c*) $C = 120°$.
(*d*) $C = 70°$. (*e*) $C = 110°$. (*f*) $C = 160°$.

9. Show that the sum of the angles of a right spherical triangle is greater than 180° and less than 360°.

9-6. Formulas relating to the right spherical triangle. Since spherical triangles having more than one right angle can be solved by inspection, we shall be concerned with right spherical triangles having only one right angle.

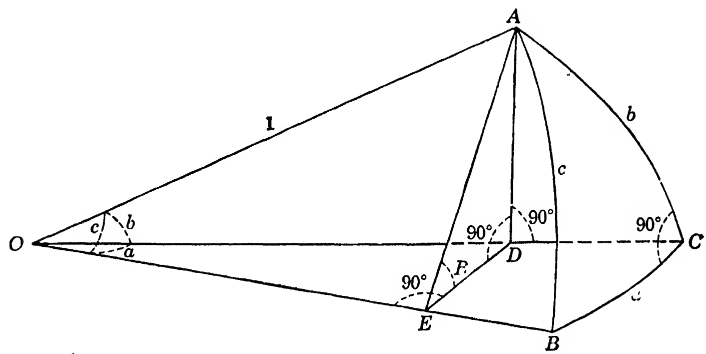


FIG. 9-4.

In this article, ten formulas relating to the right spherical triangle are derived, and in the next article simple rules for writing these formulas are given.

The solution of all cases of spherical triangles generally considered in spherical trigonometry can be solved by means of these formulas.

In Fig. 9-4 is represented a spherical pyramid that is part of a sphere having unit radius and center O. In the right spherical triangle ABC bounding the base of the pyramid, C is a right angle, and therefore the dihedral angle having edge OC is a right dihedral angle. From A, a plane is passed perpendicular to edge OB cutting the spherical pyramid in the triangle AED. Since OE is perpendicular to plane AED, it is perpendicular to lines EA and ED. Hence angle AED is the plane angle of the dihedral angle having OB as edge. Therefore angle AED is equal to

angle B. Also, plane AED is perpendicular to plane COB, since it is perpendicular to a line in the plane. Therefore line AD is perpendicular to plane OBC because it is the intersection of the two planes OAD and ADE, both of which are perpendicular to OBC. Hence the angles ADO and ADE are right angles. Each face angle of the trihedral angle O-ABC is measured by the side of the spherical triangle intercepted by it and is therefore designated by the same letter as that side.

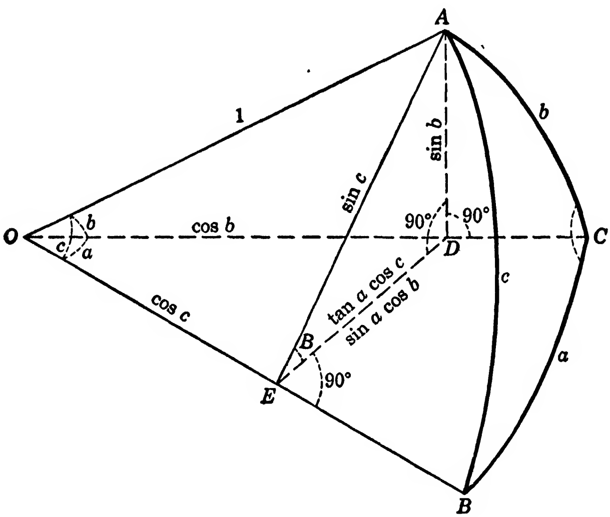


FIG. 9-5.

From Fig. 9-4 we read

$$\frac{DA}{1} = \sin b, \qquad \frac{EA}{1} = \sin c, \qquad \frac{OE}{1} = \cos c, \qquad \frac{OD}{1} = \cos b. \qquad \text{(I)}$$

Also from triangle OED, $ED/OE = \tan a$. Replacing OE in this by $\cos c$ from (1) and simplifying slightly, we have

$$ED = OE \tan a = \cos c \tan a. \qquad \text{(II)}$$

Similarly, from triangle OED,

$$ED = OD \sin a = \cos b \sin a. \qquad \text{(III)}$$

Figure 9-5 is obtained from Fig. 9-4 by enlarging it and writing on it the values of the line segments just derived.

Both values for ED, one from (II) and the other from (III) are written on ED. From the triangle AED in Fig. 9-5, we read

$$\sin B = \frac{\sin b}{\sin c}, \tag{IV}$$

$$\cos B = \frac{\tan a \cos c}{\sin c},$$

$$\tan B = \frac{\sin b}{\sin a \cos b}.$$

$$\tan a \cos c = \sin a \cos b.$$

These last four equations may be written in the following form:

$$\mathbf{\sin \mathit{b} = \sin \mathit{c} \sin \mathit{B},} \tag{1}$$
$$\mathbf{\cos \mathit{B} = \tan \mathit{a} \cot \mathit{c},} \tag{2}$$
$$\mathbf{\sin \mathit{a} = \tan \mathit{b} \cot \mathit{B},} \tag{3}$$
$$\mathbf{\cos \mathit{c} = \cos \mathit{a} \cos \mathit{b}.} \tag{4}$$

Similarly, by passing a plane through B of Fig. 9-4 perpendicular to OA and proceeding as above, we could prove the formulas

$$\mathbf{\sin \mathit{a} = \sin \mathit{c} \sin \mathit{A},} \tag{5}$$
$$\mathbf{\cos \mathit{A} = \tan \mathit{b} \cot \mathit{c},} \tag{6}$$
$$\mathbf{\sin \mathit{b} = \tan \mathit{a} \cot \mathit{A}.} \tag{7}$$

Formulas (5), (6), and (7) are the result of interchanging a and b and A and B in (1), (2), and (3), respectively. From (7) $\cot A = \sin b/\tan a$ and from (3) $\cot B = \sin a/\tan b$; multiplying these two equations member by member, we obtain

$$\cot A \cot B = \frac{\sin b}{\tan a} \times \frac{\sin a}{\tan b} = \cos b \cos a,$$

or, interchanging members and replacing $\cos b \cos a$ by $\cos c$ from (4),

$$\mathbf{\cos \mathit{c} = \cot \mathit{A} \cot \mathit{B}.} \tag{8}$$

Similarly from (2), (5), and (4), we obtain

$$\mathbf{\cos \mathit{B} = \cos \mathit{b} \sin \mathit{A},} \tag{9}$$

and from (6), (1), and (4),

$$\mathbf{\cos \mathit{A} = \cos \mathit{a} \sin \mathit{B}.} \tag{10}$$

9-7. Napier's rules. The ten formulas derived in Art. 9-6 need not be memorized, for it is easy to write them by using two rules devised by John Napier, the inventor of logarithms.

Figure 9-6 represents a right spherical triangle. Figure 9-7 contains the same letters as Fig. 9-6 except $C(= 90°)$, arranged in the same order. The bars on the letters c, B, and A mean *the complement of;* thus $\bar{B}$ means $90° - B$. *Note that the barred parts are the hypotenuse and the two angles each of which has a side along* the *hypotenuse.* The angular quantities a, b, $\bar{c}$, $\bar{A}$, $\bar{B}$

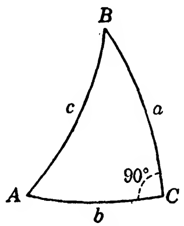


FIG. 9-6.

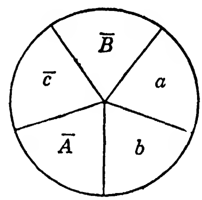


FIG. 9-7.

are called the **circular parts.** There are two circular parts contiguous with any given part and two parts that are not contiguous to it. Speaking of this given part as the **middle part,** we call the two contiguous parts the **adjacent parts,** and the two non-contiguous parts the **opposite parts.** Napier's rules may now be stated as follows:

Napier's Rule I. The sine of any middle part is equal to the product of the cosines of the opposite parts.

Napier's Rule II. The sine of any middle part is equal to the product of the tangents of the adjacent parts.

Thinking of any part as the middle part, we can write two formulas, one from each of the two rules. Considering each of the five parts in turn as middle part, we may write ten formulas, and these are found to be the ten formulas numbered (1) to (10) in Art. 9-6.*

Example. Use Napier's rules to write two formulas by using (a) b as middle part; (b) A as middle part.

* After the student has become familiar with the use of Napier's rules, he may save time by writing the desired formulas directly from the triangle on which the letters have been properly barred.

Solution. Noting that $\sin \bar{A} = \sin (90° - A) = \cos A$, $\cos \bar{A} = \cos (90° - A) = \sin A$, etc., and applying the first rule to

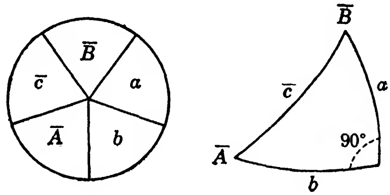


FIG. 9-8.

the parts b, $\bar{c}$, $\bar{B}$, we obtain

$$\sin b = \cos \bar{c} \cos \bar{B},$$

or

$$\sin b = \sin c \sin B. \qquad (a)$$

Applying the second rule, using parts $\bar{A}$, b, a, we obtain

$$\sin b = \tan \bar{A} \tan a = \cot A \tan a. \qquad (b)$$

Similarly, using the parts $\bar{A}$, $\bar{B}$, a and the first rule, and afterwards the parts $\bar{c}$, $\bar{A}$, b and the second rule, we obtain

$$\sin \bar{A} = \cos \bar{B} \cos a, \quad \text{or} \quad \cos A = \sin B \cos a, \qquad (c)$$

$$\sin \bar{A} = \tan \bar{c} \tan b, \quad \text{or} \quad \cos A = \cot c \tan b. \qquad (d)$$

The formulas (a), (b), (c), and (d) are, respectively, the formulas (1), (7), (10), and (6) of Art. 9-6.

EXERCISES 9-2

1. Solve each of the following right spherical triangles for the unknown part indicated:

(*a*) $a = 30°$, $b = 60°$, $c = ?$

(*b*) $c = 60°$, $a = 45°$, $B = ?$

(*c*) $a = 45°$, $B = 60°$, $c = ?$

(*d*) $a = 60°$, $B = 30°$, $A = ?$

(*e*) $c = 60°$, $A = 45°$, $b = ?$

(*f*) $A = 30°$, $B = 60°$, $a = ?$

2. Using Fig. 9-9, show that formulas (1) to (10) hold true for the case a is greater than 90°, c is greater than 90°, b is less than 90°.

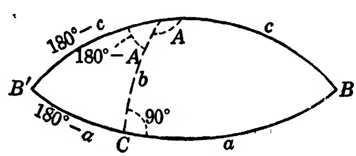


FIG. 9-9.

3. Solve each of the following right spherical triangles for the unknown part indicated:

(*a*) $a = 60°,\ b = 120°,$	$A = ?$	(*b*) $c = 135°,\ b = 120°,$	$a = ?$
(*c*) $B = 150°,\ c = 120°,$	$a = ?$	(*d*) $A = 135°,\ B = 60°,$	$c = ?$
(*e*) $a = 30°,\ B = 120°,$	$A = ?$	(*f*) $c = 120°,\ a = 135°,$	$B = ?$

4. Corresponding to each of the following formulas pertaining to a plane right triangle, write, using Napier's rules, an analogous formula pertaining to a right spherical triangle.

(*a*) $\sin A = a/c$. (*b*) $\sin B = b/c$. (*c*) $1 = \cot A \cot B$.
(*d*) $\cos A = b/c$. (*e*) $\cos B = a/c$. (*f*) $\tan A = a/b$.
(*g*) $\tan B = b/a$.

5. On Fig. 9-10 interchange A and B, also a and b. Then express the values of the line segments OD, OE, BE, BD, DE in terms of a, b, c,

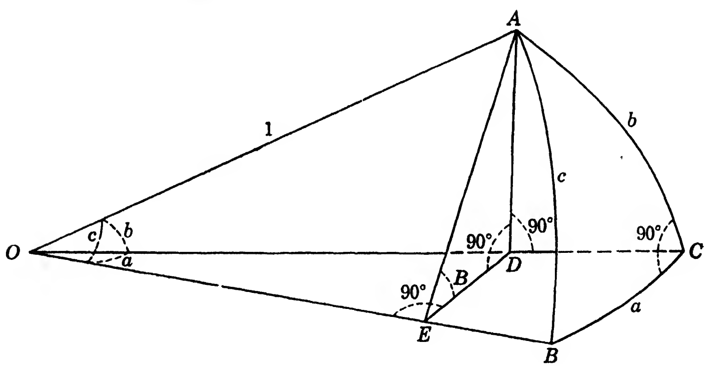


FIG. 9-10.

and write each of these line values on the figure. Equate two values of DE to obtain formula (4), and apply the definitions of the trigonometric functions to triangle BDE to obtain formulas (5), (6), and (7).

6. Using formula (4), show that the hypotenuse of a right spherical triangle is less than or greater than 90°, according as the two legs lie in the same quadrant or in different quadrants.

7. Using formula (10), show that in a right spherical triangle each leg and the opposite angle are of the same quadrant.

8. Use Napier's rules to write a formula involving the following, taking c as unknown part,

(a) c, B, A. (b) c, B, a. (c) c, B, b.

9. Use Napier's rules to write three formulas, each involving a and b.

10. Prove that $\tan A = \dfrac{\sin a}{\tan b \cos c}$.

11. Prove that $\cos A = \dfrac{\sin b \cos a}{\sin c}$.

9-8. Two important rules. In what follows it will be convenient to speak of an angle of the first quadrant or of the second quadrant. An angle is said to be of the first, second, third, or fourth quadrant according as its terminal side falls in the first, second, third, or fourth quadrant when laid off in the usual manner relative to rectangular coordinate axes.

From formula (10) of Art. 9-6, namely,

$$\cos A = \cos a \sin B,$$

it follows that $\cos A$ and $\cos a$ must have the same sign since $\sin B$ is positive in all cases. Hence both A and a must be less than 90°, or both must be greater than 90°. Formula (9) may be used to show that B and b must be of the same quadrant. The following rule expresses the relation.

Rule I. In a right spherical triangle an oblique angle and the side opposite are of the same quadrant.

From formula (4), namely,

$$\cos c = \cos a \cos b,$$

it appears that the product $\cos a \cos b$ must be positive when c is less than 90°; therefore $\cos a$ and $\cos b$ must have the same sign, and for that reason a and b are both of the first quadrant or both of the second quadrant. From the same formula it appears that $\cos a \cos b$ must be negative when c is greater than 90°; therefore $\cos a$ and $\cos b$ must have opposite signs, and a and b are of different quadrants. The following rule expresses the relation.

Rule II. When the hypotenuse of a right spherical triangle is less than 90°, the two legs are of the same quadrant; when the hypotenuse is greater than 90°, one leg is of the first quadrant and the other of the second.

Rules I and II enable the computer to tell the quadrant of an angle found from its sine or its cosecant.

EXERCISES 9-3

State the quadrant of each of the unknown parts in each of the right spherical triangles indicated in the following diagram:

	a	b	c	A	B
1	30°	40°			
2	30°		120°		
3	120°				50°
4		140°	75°		
5				120°	130°
6		35°		100°	
7			100°	100°	
8			60°		60°

9-9. Solution of right spherical triangles. When two parts of a right spherical triangle in addition to the right angle are given, the remaining parts can be computed from formulas obtained by using Napier's rules. In solving the triangle it will be found advantageous to proceed as follows:

a. Draw a right spherical triangle lettered in the conventional way and encircle the given parts.

b. Write a formula for each unknown part by applying Napier's rules. *Each formula should contain the unknown part and both of the given parts.* Then write a check formula connecting the three required parts.

c. Make a form.

d. Fill in the blank spaces of the form.

Example. Solve the right spherical triangle in which

$$a = 66°59', \qquad b = 156°34'.$$

Solution. Figures 9-11 and 9-12 display the circular parts of a right spherical triangle, the known parts a, b being encircled.

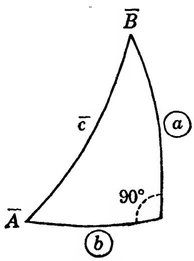


FIG. 9-11.

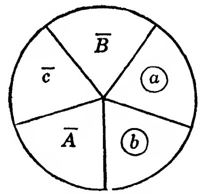


FIG. 9-12.

Using Napier's rules, in connection with Fig. 10, we write

$\sin b = \tan a \cot A,$	or	$\cot A = \sin b \cot a,$		(a)
$\sin a = \tan b \cot B,$	or	$\cot B = \sin a \cot b,$		(b)
$\cos c = \cos a \cos b,$				(c)
$\cos c = \cot A \cot B.$				(d)

The symbols "l sin," "l cot," etc., written in any line of a form mean log sine of the angle at the left of the line, log cotangent of that angle, etc. For convenience the negative part -10 of the characteristic will be omitted in the forms.

The symbol $(-)$ written before a logarithm in any form calls attention to the fact that the antilogarithm of that logarithm is negative. Hence an odd number of symbols $(-)$ appearing in a column used to evaluate a product by logarithms will indicate that the product is negative. An even number of symbols $(-)$ will indicate a positive product.

The computation of the unknown parts from the formulas (a), (b), (c), and the check by (d) follow.

	(a) and *check*		(b)		(c)	
$a = 66°59'$	l cot	9.6282	l sin	9.9640	l cos	9.5922
$b = 156°34'$	l sin	9.5995	l cot	$(-)$ 0.3631	l cos	$(-)$ 9.9626
$A =$ **80°25′**	l cot	9.2277				
$B =$ **154°47′**	l cot	$(-)$ 0.3271	l cot	$(-)$ 0.3271		
$c =$ **111°1′**	l cos	$(-)$ 9.5548			l cos	$(-)$ 9.5548

Observe that the results obtained by adding l cot A to l cot B to get l cos c check only the logarithms of the computed parts. Errors made in finding A, B, and c from associated logarithms would not affect the check.

EXERCISES 9-4

Solve the following right spherical triangles:

1. $a = 10°32'$, $B = 12°3'$.

2. $c = 46°40'$, $B = 20°50'$.

3. $a = 118°54'$, $B = 12°19'$.

4. $a = 43°27'$, $c = 60°24'$.

5. $b = 48°36'$, $c = 69°42'$.

6. $a = 168°13'$, $c = 150°9'$.

7. $c = 112°48'$, $B = 56°11'$.

8. $c = 32°34'$, $A = 44°44'$.

9. $A = 116°31'$, $B = 116°43'$.

10. $A = 54°54'$, $c = 69°25'$.

11. $c = 55°9'$, $a = 22°15'$.

12. $a = 36°27'$, $b = 43°32'$.

13. $a = 29°46'$, $B = 137°24'$.

14. $a = 144°27'$, $b = 32°8'$.

15. $b = 36°27'$, $a = 43°32'$.

16. $A = 63°15'$, $B = 135°33'$.

17. $A = 67°54'$, $B = 99°57'$.

18. $b = 22°15'$, $c = 55°9'$.

19. $a = 118°30'$, $B = 95°36'$.

20. $b = 92°47'$, $A = 50°2'$.

21. If angle A of a right spnerical triangle is 28°, what is the maximum size of angle B?

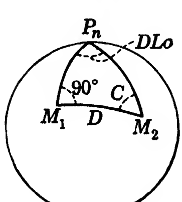


FIG. 9-13.

22. A ship leaves point M_1 in Fig. 9-13 sailing due east and follows a great-circle track to a point M_2. If M_1 is in latitude 40°30′ N., longitude 75° W. and if M_2 is in longitude 60° W., find the distance D traveled, the latitude of M_2, and the course angle C at M_2.

Hint. The angle *DLo* at the north pole P_n is the difference in the longitudes of the two points M_1 and M_2. The distances from the points M_1 and M_2 to P_n are, respectively, the complements of the latitudes of these points.

23. In the spherical triangle ABC (Fig. 9-14), p is the arc of a great circle perpendicular to side c. Write an expression for B in terms of A, a, and b.

Hint. Find two values of p and equate them.

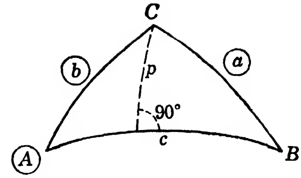


FIG. 9-14.

24. If in the triangle ABC of Exercise 23, $A = 40°10'$, $a = 46°20'$, and $b = 64°50'$, find B.

25. All lines in Fig. 9-15 represent arcs of great circles. Find all unknown parts, thus solving a spherical triangle for which two angles and the included side are given.

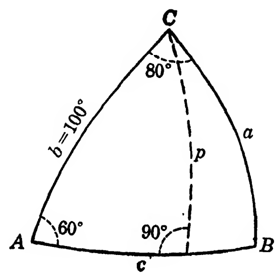


FIG. 9-15.

9-10. The ambiguous case. When the given parts are a side and the angle opposite, two solutions are obtained. In such cases each unknown part is found from the sine and hence may be chosen either in the first quadrant or in the second quadrant; that is, in the case of each unknown an angle and its supplement must be written. If A and a represent the given parts and C the right angle, the two triangles will form a lune as indicated in Fig. 9-16; for in this figure B' appears as $180° - B$, c' as $180° - c$, and b' as $180° - b$.

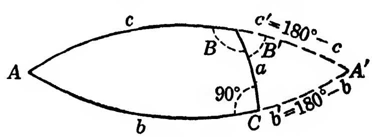


FIG. 9-16.

The solution of the following example will illustrate the method of finding a double solution when it exists.

Example. Solve the right spherical triangle in which

$$a = 46°45', \qquad A \doteq 59°12'.$$

Solution. Using Napier's rules in connection with Fig. 9-17, we obtain

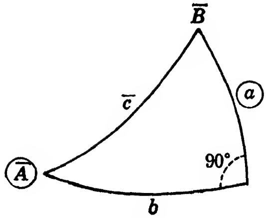


FIG. 9-17.

$$\begin{aligned} \sin c &= \sin a \csc A, && (a)\\ \sin B &= \sec a \cos A, && (b)\\ \sin b &= \tan a \cot A, && (c)\\ \sin b &= \sin c \sin B. && \textit{Check} \end{aligned}$$

The solution is displayed below.

	(*a*) and (*check*)	(*b*)	(*c*)
$a = 46°45'$	l sin 9.8624	col cos 0.1642	l tan 0.0626
$A = 59°12'$	col sin 0.0660	l cos 9.7093	l cot 9.7753
$c_1 = \mathbf{58°0'}$	l sin 9.9284		
$c_2 = \mathbf{122°0'}$			
$B_1 = \mathbf{48°22'}$	l sin 9.8735	l sin 9.8735	
$B_2 = \mathbf{131°38'}$			
$b_1 = \mathbf{39°19'}$	l sin 9.8019		l sin 9.8019
$b_2 = \mathbf{140°41'}$			

The six answers were grouped to obtain the solutions b_1, c_1, B_1, and b_2, c_2, B_2 by using Rules I and II of Art. 9-8.

EXERCISES 9-5

Solve the following right spherical triangles:

1. $b = 35°44'$, $B = 37°28'$.

2. $a = 77°21'$, $A = 83°56'$.

3. $b = 129°33'$, $B = 104°59'$.

4. $a = 160°$, $A = 150°$.

5. $b = 21°39'$, $B = 42°10'$.

6. $b = 42°18'$, $B = 46°15'$.

7. Apply Napier's rules to Fig. 9-17 to obtain a formula involving the known parts a, A, and the unknown part b. From this formula show that there may be no solution. Discuss the case that arises when a and A are supplementary.

Solve the following right spherical triangles:

8. $b = 42°18'$, $B = 42°18'$.

9. $a = 20°10'$, $A = 115°20'$.

9-11. Polar triangles. The poles of a great circle on a sphere are the points where a perpendicular to the plane of the great

circle through its center pierces the surface of the sphere. To obtain the polar triangle of a spherical triangle ABC, construct three great circles on the sphere having their poles at A, B, and C. Two arcs, one having B as pole and the other C as pole, intersect in two points on opposite sides of arc BC. Denote by A' the point that lies on the same side of the great circle through BC as A. Locate B' and C' by an analogous procedure. Then triangle $A'B'C'$ is the polar of triangle ABC. Figure 9-18 indicates the relations.

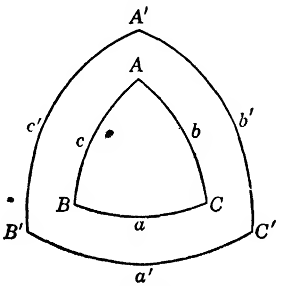


FIG. 9-18a.

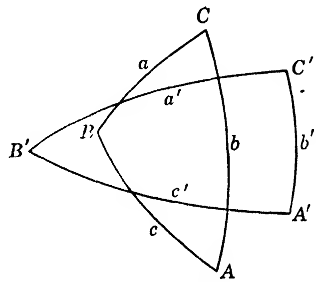


FIG. 9-18b.

The following theorems from solid geometry are important:

1. *If $A'B'C'$ represents the polar triangle of spherical triangle ABC, then ABC is the polar triangle of $A'B'C'$.*

2. *An angle of any spherical triangle is the supplement of the opposite side in the polar triangle.*

In accordance with Theorem 2, we have the following relations between the sides and angles represented in Fig. 9-18:

$$\left.\begin{aligned} A' &= 180^\circ - a, \qquad & A &= 180^\circ - a', \\ B' &= 180^\circ - b, \qquad & B &= 180^\circ - b', \\ C' &= 180^\circ - c, \qquad & C &= 180^\circ - c'. \end{aligned}\right\} \tag{11}$$

If, in an equation containing the quantities a, b, c, A, B, C, these quantities are replaced by their values in terms of a', b', c', A', B', C', from (11), a new equation having reference to the polar triangle is obtained. The relations (11) will be used in the next article to solve a spherical triangle having a side equal to 90°.

EXERCISES 9-6

1. Use relations (11) to find the parts of the polar triangle of each of the following spherical triangles:

(a) $A = 135°59.1'$, $B = 100°10.1'$, $C = 98°43.3'$, $c = 90°$,

$$a = 135°20', \qquad b = 98°31.5'.$$

(b) $a = 54°16.0'$, $b = 114°47.0'$, $C = 70°35.9'$, $c = 90°$,

$$A = 49°57.9', \qquad B = 121°5.5'.$$

(c) $a = 116°35.6'$, $b = 105°14.8'$, $c = 43°17.2'$, $A = 112°47.4'$, $B = 84°6.7'$, $C = 44°59.1'$.
(d) $a = 136°19.6'$, $b = 43°18.5'$, $c = 114°43.3'$, $A = 132°15.3'$, $B = 47°19.5'$, $C = 76°48.4'$.

2. For each of the following formulas, write a new formula having reference to the polar triangle:

(a) $\sin a = \sin c \sin A$.
(b) $\tan b = \tan c \cos A$.
(c) $\tan a = \sin b \tan A$.
(d) $\cos c = \cos b \cos a$.
(e) $\sin b = \sin c \sin B$.
(f) $\cos a = \cos b \cos c + \sin b \sin c \cos A$.
(g) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$.
(h) $\dfrac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = \dfrac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a + b)}$.
(i) $\dfrac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} = \dfrac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a - b)}$.

3. For each of the following triangles find the known parts of the polar triangle. Solve these polar triangles.

(a) $c = 90°$, $a = 122°48.2'$, $B = 21°35.4'$.
(b) $c = 90°$, $a = 49°30.0'$, $B = 65°36.2'$.

9-12. Quadrantal triangles. A spherical triangle having a side equal to 90° is called a **quadrantal triangle.** Evidently the polar triangle of a quadrantal triangle is a right spherical triangle. Hence this polar triangle can be solved in the usual way, and the unknown parts of the quadrantal triangle can then be obtained by using relations (11).

Example. Solve the spherical triangle in which $c = 90°$, $A = 115°38'$, $b = 139°58'$.

Solution. Using (11) of Art. 9-11, we obtain for the polar triangle $C' = 180° - c = 90°$, $a' = 180° - A = 64°22'$,

$$B' = 180° - b = 40°2'.$$

The solution of the polar triangle follows:

$a' = 64°22'$	l cot 9.6811	l sin 9.9550	l cos 9.6361
$B' = 40°2'$	l cos 9.8841	l tan 9.9243	l sin 9.8084
$c' = 69°49'$	l cot 9.5652		
$b' = 37°9'$	l tan 9.8793	l tan 9.8793	
$A' = 73°50'$	l cos 9.4445		l cos 9.4445

Using equations (11) again, we obtain $C = \mathbf{180° - c' = 110°11'}$, $B = \mathbf{180° - b' = 142°51'}$, $a = \mathbf{180° - A' = 106°10'}$.

EXERCISES 9-7

Solve the following right spherical triangles and then use (11) to obtain the solution of the polar triangle of each:

1. $a = 115°6'$,
$b = 123°14'$.

2. $a = 112°43'$,
$c = 85°10'$.

Solve the following quadrantal triangles:

3. $B = 117°54'$,
$a = 95°42'$,
$c = 90°$.

4. $A = 153°16'$,
$b = 19°3'$,
$c = 90°$.

5. $B = 69°45'$,
$A = 94°40'$,
$c = 90°$.

6. $b = 159°33'$,
$a = 95°18'$,
$c = 90°$.

7. In Fig. 9-19, $a = 18°12'$, $B = 74°45'$, $c = 90°$. Solve the right triangle ACD, and from it deduce the solution of the quadrantal triangle ABC.

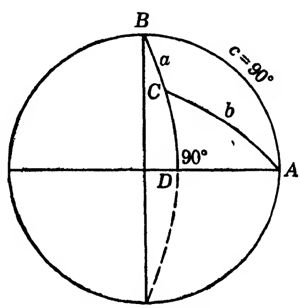


FIG. 9-19.

9-13. The solution of the oblique triangle. We have seen that any right spherical triangle can be solved by the use of Napier's rules. An oblique spherical triangle can be solved by dividing it into two right triangles and then using Napier's rules to solve each of them. When the given parts are two sides and the included angle, drop the perpendicular from the vertex of an unknown angle to the opposite side. An example will serve to indicate the method.

Example. Solve the spherical triangle in which $a = 88°24'$, $b = 56°48'$, $C = 128°16'$.

Solution. Figure 9-20 represents a triangle with the given parts encircled and with the arc AD drawn perpendicular to the

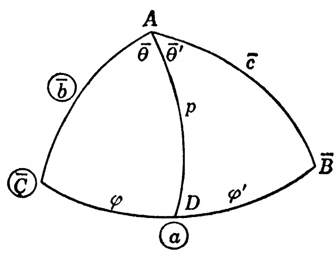


FIG. 9-20.

side BC. Applying Napier's rules to the right triangle ACD, we obtain the formulas

$$\tan \varphi = \tan b \cos C \tag{12}$$
$$\cot \theta = \cos b \tan C \tag{13}$$
$$\sin p = \sin b \sin C \tag{14}$$
$$\sin p = \cot \theta \tan \varphi \ (check) \tag{15}$$

The solution of the right triangle ADC by using (12), (13), (14), and (15) follows.

	(12)(15)	(13)	(14)
$b = 56°48'$	l tan 0.1842	l cos 9.7384	l sin 9.9226
$C = 128°16'$	l cos (−)9.7920	l tan (−)0.1030	l sin 9.8949
$\varphi = 136°44'$	l tan (−)9.9762		
$\theta = 124°46'$	l cot (−)9.8414	l cot (−)9.8414	
$p = 138°55'$	l sin 9.8176		l sin 9.8175

After the first right triangle has been solved, the figure should be drawn showing the perpendicular falling inside or outside the

triangle according as φ is less than or greater than the side along which it lies.

Since φ is greater than a, the point D falls outside the arc $\bar{C}B$ extended as indicated in Fig. 9-21. In the triangle BDA the arcs

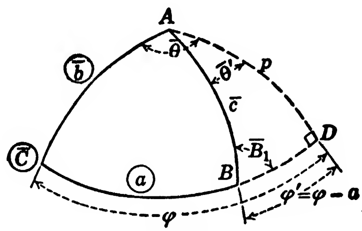


FIG. 9-21.

p and $\varphi' = \varphi - a$ are known. Applying Napier's rules to triangle BDA, we obtain the following formulas:

$$\cot B_1 = \cot p \sin \varphi' \tag{16}$$
$$\cot \theta' = \sin p \cot \varphi' \tag{17}$$
$$\cos c = \cos p \cos \varphi' \tag{18}$$
$$(\textit{check}) \cos c = \cot \theta \cot B_1 \tag{19}$$

The solution of the triangle BDA follows:

	(16)(19)	(17)	(18)
$p = 138°55'$	l cot (−) 0.0595	l sin 9.8716	l cos (−) 9.8772
$\varphi' = (\varphi - a) = 48°10'$	l sin 9.8722	l cot 9.9519	l cos 9.8241
$B_1 = 130°31'$	l cot (−) 9.9317		
$\theta' = 59°32'$	l cot 9.7695	l cot 9.7695	
$c = 120°10'$	l cos 9.7012		l cos (−) 9.7013

Using Fig. 9-21 and the quantities obtained in the solution, we have $B = 180° - B_1 =$ **49°27′**, $A = \theta - \theta' =$ **65°14′**,

$C =$ **120°10′**.

EXERCISES 9-8

Solve the following spherical triangles by the method of this article:

1. $a = 88°24'$, $b = 56°48'$, $C = 128°16'$.

2. $a = 88°37'$, $c = 125°18'$, $B = 102°16'$.

3. $b = 120°30'$, $c = 70°20'$, $A = 50°10'$.

4. $a = 86°18'$, $b = 45°36'$, $C = 120°46'$.

5. $a = 76°24'$, $b = 58°19'$, $C = 116°30'$.

6. $b = 132°17'$, $c = 78°15'$, $A = 40°20'$.

Solve the following triangles by solving the polar triangle:

7. $A = 120°10'$, $B = 100°20$, $c = 30°5'$.

8. $A = 27°22'$, $C = 91°26'$, $b = 120°18'$.

Solve the following spherical triangles by the method of this article:

9. $a = 40°6'$, $b = 118°22'$, $A = 29°43'$.

10. $a = 150°57'$, $b = 134°15'$, $A = 144°22'$.

11. $a = 128°15'$, $b = 129°20'$, $A = 130°25'$.

12. $a = 52°45'$, $c = 71°12'$, $A = 46°22'$.

13. Solve each of the following triangles by solving its polar triangle:

(*a*) $c = 80°13'$, $C = 78°15'$, $B = 75°17'$.

(*b*) $a = 115°13'$, $A = 120°43'$, $B = 116°38'$.

CHAPTER 10

ELEMENTARY APPLICATIONS

10-1. The terrestrial sphere. The earth is considered here as a sphere about 7917 statute miles in diameter. Actually it is elliptical, its shortest diameter through the poles being about 27 statute miles shorter than the equatorial diameter.

The earth revolves about a line through its center called its **axis.** The points in which it cuts the surface are called **poles.** Figure 10-1 represents the earth, P_nP_s its **axis,** P_n the **north pole,** and P_s the **south pole.**

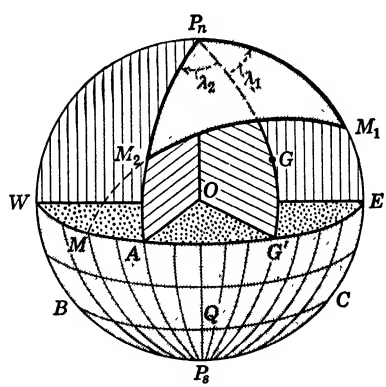


FIG. 10-1.

A plane through the center of a sphere cuts it in a **great circle.** Any plane intersecting the sphere but not passing through its center cuts it in a **small circle.** In Fig. 10-1, P_nEP_sW and WME represent great circles and CQB a small circle.

The **equator** is the great circle cut out by the plane perpendicular to the axis of the earth at its center. $WMAE$ in Fig. 10-1 represents the equator.

A **parallel of latitude,** or briefly a parallel, is a small circle cut out by a plane parallel to the plane of the equator. CQB in Fig. 10-1 represents a parallel of latitude.

A **meridian** on the earth is a great circle passing through the north pole and the south pole. P_nGP_s and P_nEP_s in Fig. 10-1 represent meridians.

The latitude of a place on the earth is its angular distance from the equator. It is measured from the equator along a

meridian and is less than 90°. In Fig. 10-1 the angular measure of arc AM_2, that is, angular $AOM_2 = L$, the latitude of point M_2. In general, north latitude is considered positive, south latitude negative.

The longitude of a point on the earth is the angle at either pole between the meridian passing through the point and some fixed meridian known as the prime meridian. It is measured from 0° to 180° east or west of the prime meridian. The meridian of Greenwich, England, is the prime meridian not only for American and English navigators, but also for those of many

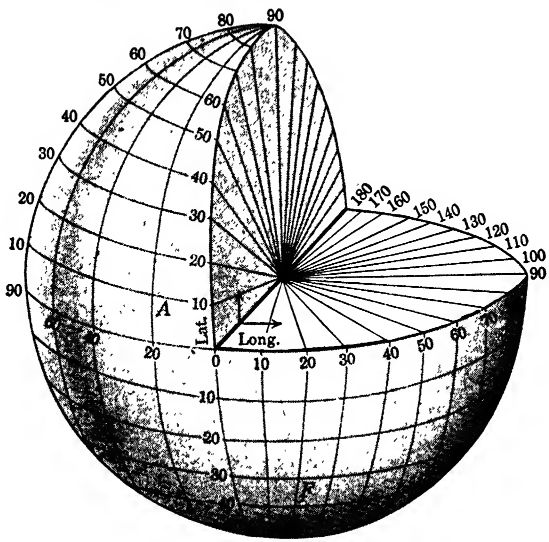


FIG. 10-2.

other nations. If $P_nGG'P_s$ in Fig. 10-1 represents the meridian of Greenwich, then angle λ_1 = angle GP_nM_1 = angle $G'OE$ is the longitude of point M_1.

Figure 10-2 represents the earth with one-quarter cut away. The numbers along the equator represent longitudes and those along the meridian represent latitudes.

A **nautical mile** is 6080.27 ft. Laid along a great circle on a sphere the size of the earth, it would subtend at its center an angle of 1′. Thus the number of minutes in the arc of a great circle on the earth is, for practical purposes, the number of nautical miles in its length. For example, the distance between

two points on a meridian 50°25′48″ apart would be

$$50 \times 60 + 25 + \tfrac{48}{60} = 3025.8 \text{ nautical miles.}$$

The circumference of the earth, containing 360°, is 360 × 60 or 21,600 nautical miles. The radius R of the earth, since circumference = $2\pi R$, is 21,600 ÷ 2π or 3437.7 nautical miles.

10-2. The terrestrial triangle. The triangle $M_1P_nM_2$ of Fig. 10-1 is used in connection with problems relating to distances and angles on the earth and is called the **terrestrial triangle.** Arc M_1M_2 represents the distance along the great-circle track from M_1 to M_2, and the angle $M_2M_1P_n$, or C_n, gives the initial direction of the track. The angle of departure

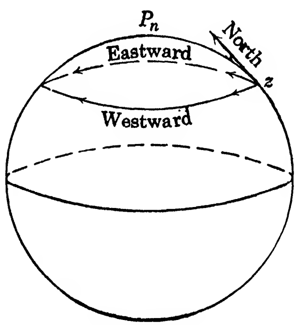


FIG. 10-3.

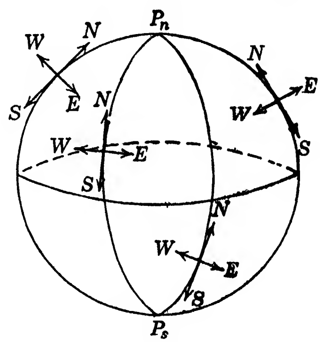


FIG. 10-4.

$P_nM_1M_2$ measured from the north around through the east from 0° to 360° is called the initial course C_n. The angle $M_1P_nM_2$ is the difference in longitude DLo (or $\lambda_2 - \lambda_1$) between M_1 and M_2. Arc P_nM_1 is $90° - L_1$, and arc P_nM_2 is $90° - L_2$ where L_1 and L_2 refer to the latitudes of M_1 and M_2.

Observe that when λ_1 is the longitude of a point east of Greenwich and λ_2 that of a point west of it, DLo is the sum of $\lambda_1 + \lambda_2$ or $360° - (\lambda_1 + \lambda_2)$ according as $\lambda_1 + \lambda_2$ is less than or more than 180°.

It is important to make the correct association of directions with lines on a diagram like Fig. 10-3. For a person situated on the Northern Hemisphere of the earth at a point such as z in Fig. 10-3, north is along the tangent to the meridian away from the equator; for a person standing at z facing north, east is on his right, west is on his left, and south is opposite to the direction

in which he is facing. Figure 10-4 indicates directions at four positions on the earth.

EXERCISES 10-1

1. Define:

(*a*) Axis of the earth.	(*b*) Poles of the earth.
(*c*) Great circle of a sphere.	(*d*) Small circle of a sphere.
(*e*) Equator of the earth.	(*f*) Meridian.
(*g*) Parallel.	(*h*) Prime meridian.
(*i*) Terrestrial triangle.	(*j*) Nautical mile.

2. Find the distance in miles between two points on the equator having respective longitudes (*a*) 20° W., 30° W.; (*b*) 0° W., 90° W.; (*c*) 20° W., 30° E.

3. Find the difference in longitude (*DLo*) of two places, if the longitude of one is 40°28′ W. and that of the other is 23°38′ E.

4. Two places are on the parallel 30° N. The longitude of one place is 35° W. and that of the other is 65° W. Find the distance between them measured along the parallel.

5. What is the latitude of the parallel that is one-half the equator?

6. Find the length in miles of the parallel 30° N.

7. The difference in latitude and longitude between New York and Paris is 8°10′ and 76°15′, respectively, and New York is farther west and south than Paris. Find the latitude and longitude of Paris if New York is in Lat. 40°42′ N., Long. 73°55′ W.

8. Find the polar distance of Moscow, $L = 55°50'$ N., $\lambda = 37°35'$ E.

9. Washington, D.C., is in Lat. 38°52′ N. and Long. 77°0′ W. What are the latitude and longitude of a place diametrically opposite Washington?

10. The following refer to the terrestrial triangle $M_1P_nM_2$ in Fig. 10-1:

(*a*) What angle is the initial course angle?

(*b*) What is the length of the side that lies opposite the initial course angle?

(*c*) What is the angle at the pole called?

(*d*) What is the length of the side lying opposite the angle at the pole?

11. A vessel steams westward along a great-circle track departing from a place in Lat. 27° N., Long. 15° W. If her initial course angle is 123°, what is her initial true course?

12. Each line in the following table refers to a great-circle voyage. In each case find the true initial course C_n.

	Latitude of place of departure	Direction of sailing	Initial course angle
(a)	$L = 27°$ N.	Westward	27°
(b)	$L = 34°$ S.	Westward	123°
(c)	$L = 64°$ N.	Eastward	63°
(d)	$L = 19°$ S.	Eastward	115°

13. A vessel departs from a place A in Lat. 62° S. and steams eastward along a great circle track to B. If in the terrestrial triangle involved, the angle opposite the side through A is 64°, what is the true course of arrival?

14. A ship sails from a point in Lat. 40° N., Long. 28° W., to a point in Lat. 50° N., Long. 50° W. Draw the terrestrial triangle associated with this trip. Name the parts of this triangle. What are the known parts?

10-3. Parallels of latitude. In Fig. 10-5, C represents the center of the earth, P_n the north pole, AB an arc on the equator, and DE an arc of a small circle in latitude L cut out by a plane DEF parallel to the plane of the equator. From the figure it appears that angle ACB = angle DFE = angle DP_nE is the difference in longitude DLo between points A and B or between D and E. From sector ACB,

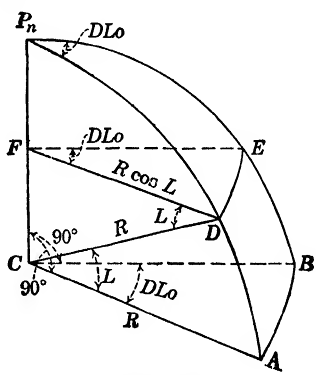


FIG. 10-5.

$$(AB)_n = R(DLo)_r, \qquad (1)$$

where $(AB)_n$ denotes arc AB in nautical miles, R the radius of the earth in nautical miles, and $(DLo)_r$ the difference in longitude in radians. But numerically

$$(AB)_n = (AB)' = (DLo)',$$

where the symbol $'$ indicates that the quantity is measured in minutes.

Hence numerically

$$(DLo)' = R(DLo)_r. \qquad (2)$$

Also from sector *DFE*

$$(DE)_n = R(\cos L)(DLo)_r,$$

where $(DE)_n$ denotes arc *DE* in nautical miles. Substituting the value of $R(DLo)_r$ from (2) in this equation, we get

$$(DE)_n = (\cos L)(DLo)'. \qquad (3)$$

10-4. Plane sailing. The path of a ship intersecting at the same angle all the meridians which it crosses is called a **rhumb line.** All rhumb lines except parallels of latitude are called **loxodromic curves.** Such a curve when sufficiently prolonged spirals about a pole but does not reach it.

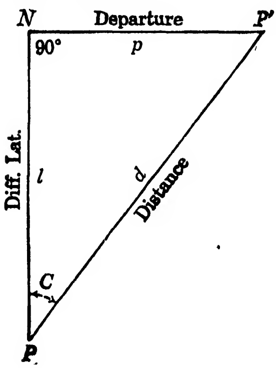


FIG. 10-6.

In Fig. 10-6, *PP'* represents a comparatively short distance along a rhumb line which cuts meridian *PN* at angle *C*. *NP'* represents part of a parallel of latitude. The lengths of $PP' = d$, $PN = l$, and $NP' = p$ are called, respectively, the **distance,** the **difference in latitude,** and the **departure.** For comparatively short distances the triangle *PNP'* is considered as a plane triangle and the following formulas are read from it:

$$l = d \cos C, \qquad p = d \sin C. \qquad (4)$$

10-5. Middle-latitude sailing. Since difference in latitude l is along a meridian, the number of nautical miles in l is the number of minutes in the difference in latitude between P and P'. Formula (3) shows that departure p must be multiplied by sec L to get *DLo*. Since L is a variable between P and P', an approximation to *DLo* in minutes is obtained by multiplying departure p by the secant of the mid-latitude $(\frac{1}{2})(Lat.\ P + Lat.\ P')$. These relations are expressed by the following formulas:

$$(\text{Diff. lat.})' = d \cos C,$$
$$(DLo)' = d \sin C \sec \tfrac{1}{2}(Lat.\ P + Lat.\ P'), \qquad (5)$$

where d is in miles. Observe that the first formula in (5) is exact, whereas the second is approximate. This method of converting departure to difference in longitude is called **middle latitude sailing.**

Example 1. An airplane flies 200 miles northeast from Annapolis, Lat. 38°59′ N., Long. 76°29′ W. Find the difference in latitude and the departure. Also find the latitude and longitude of the place reached.

Solution. Using formulas (4) we obtain

$$l = 200 \cos 45° = \mathbf{141.4\ miles}, \qquad p = 200 \sin 45° = \mathbf{141.4\ miles}. \qquad (a)$$

Hence the change in latitude is 141.4′ = 2°21.4′ and the required latitude is (38°59′ + 2°21.4′) N. = **41°20.4′ N.** Using the second formula of (5), we have

$$DLo = 200' \sin 45° \sec [38°59' + \tfrac{1}{2}(2°21.4')] = 188.5' = 3°8.5'.$$

Hence the required longitude is

$$(76°29' - 3°8.5')\ \text{W.} = \mathbf{73°20.5'\ W.}$$

Example 2. By mid-latitude sailing determine the true course and distance from P, $L = 12°17'$ S., $\lambda = 138°14'$ W., and P', $L' = 22°57'$ S., $\lambda' = 88°51'$ W.

Solution. We first derive a formula for C by dividing the second formula of (5) by the first, member by member, to obtain after slight simplification

$$\tan C = \frac{DLo \cos \frac{1}{2}(L + L')}{\text{diff. lat.}} = \frac{(DLo) \cos L_m}{l}. \qquad (a)$$

We then find d by using the first formula of (5), namely,

$$d = (\text{diff. lat.}) \cos C = l \sec C. \qquad (b)$$

The following form contains the solution:

		(a)	(b)
DLo	= 2963′ E.	log 3.4717	
$\frac{1}{2}(L + L')$	= 17°17′ S.	cos 9.9791 − 10	
$l = L' - L$	= 640′ S.	colog 7.1938 − 10	log 2.8062
C'	= 77°14′ E.	tan 0.6446	sec 0.6556
C_n	= **102.8°**		
d	= **2896 miles**		log 3.4818

EXERCISES 10-2

1. If a ship sails on a course of 42° for 190 miles, what are the departure and difference in latitude?

2. If a ship sails a course of 19° for 201.85 miles, what is the departure?

3. A ship asks bearings from two radio stations A and B. A reports the ship's bearing 82° (Navy compass) and B reports 127°. Station B is known to be 127 nautical miles from A on bearing 58° from A. Find the difference in latitude and departure of the ship from A.

In solving the following problems use formula (5):

4. A ship steams due west 120.5 miles in latitude 39°. Find the change in its longitude.

5. A ship in Lat. 47°30′ N. steams directly east until it has made good a difference in longitude of 2°30′. Find the departure.

6. A ship at point M_1, $L = 41°30'$ N., $\lambda = 59°47'$ W., steams on course 147° for 290 miles. Find the latitude and longitude of the point of arrival.

7. A ship leaves a point M_1, $L_1 = 43°19'$ N., $\lambda_1 = 17°42'$ W. and arrives at point M_2, $L_2 = 41°13'$ N., $\lambda_2 = 21°14'$ W. Find the course and distance for a rhumb-line track.

8. Find the course and distance on a rhumb-line track from a point in Lat. 34°48.1′ N., Long. 22°14.2′ W. to a point in Lat. 37°40′ N., Long. 25°40′ W.

9. (*a*) If the difference of longitude of two places A and B on the earth is 50° and their latitudes are 30°, find the distance AB measured on the equal latitude circle.

(*b*) What is the distance AB measured on a great circle? The radius of the earth is approximately 3960 land miles.

10. Two points A and B are the ends of a 500-land-mile arc of a small circle in latitude 36° N. Find the difference in their longitudes. If A_1 and B_1 are both in latitude 36° N. and the arc of a great circle connecting them is 500 land miles long, what is the difference in their longitudes? Assume the radius of the earth to be 3960 land miles.

10-6. The Mercator chart. In steaming a short distance, a ship generally follows a rhumb line for the convenience of maintaining a constant course. For added convenience navigators use freely a chart on which any rhumb line will appear as a straight line. Such a chart is called a **Mercator chart.**

On a Mercator chart the meridians appear as a set of parallel lines spaced at equal distances for equal differences in longitude; the parallels of latitude appear as a set of parallel lines perpendicular to the first set. Since the meridians are represented by parallel lines and a rhumb line must cut them at the same angle, the rhumb line must appear as a straight line on the chart.

In Fig. 10-7 the length $X'Y'$ represents the length XY on the equator, and $A'B'$ represents the arc AB of a parallel of latitude. In accordance with formula (3) arc AB = arc $XY \cos L$; and, since $A'B' = X'Y'$, it is apparent that arc AB appears on the chart expanded to $1/\cos L = \sec L$ times its natural size. Since

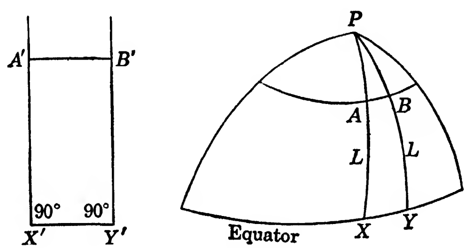


FIG. 10-7.

the parallels of latitude are expanded in the ratio sec L, the meridians near each parallel must be expanded in the same ratio to avoid local distortion. The greater the latitude the greater the distortion; for as L increases so does sec L. However, since the ratio of expansion is always sec L, the length d of any short part of a rhumb line will be approximately equal to the line segment of length d_m representing this part on the map multiplied by the cosine of the mid-latitude for the segment. In symbols

$$d = d_m \cos (\text{mid-lat.}). \tag{6}$$

If B in Fig. 10-7 is in latitude L and the earth is assumed spherical in shape, the distance $Y'B'$ on the map would be, to some scale, $(21{,}600/2\pi) \log \tan (45° + L)$ miles. Because of the fact that the meridians are slightly elliptical, this formula cannot be used for large distances.

The scale for the maps shown (see Fig. 10-8) is such that $\frac{1}{2}$ in. is assigned to each degree of longitude (or of latitude at the equator). Hence any length on the map can be changed to minutes, and therefore to miles by multiplying its length in inches by 120, or

by *laying it off along the horizontal longitude scale and reading the corresponding number of degrees and minutes directly.*

The essential facts may be summarized as follows:

When the length d_m of any line is found in minutes of the longitude scale, *the corresponding true length d may be obtained by using*

$$d = d_m \cos \text{(mid-lat.), (approx.).} \tag{6}$$

Also the latitudes of the ends of the line may be read from the chart and used in the first of formulas (5) slightly transformed to read

$$d = (L_2 - L_1)' \sec C. \tag{7}$$

Observe that $L_2 - L_1$ must be expressed in minutes and that C, the course angle, may be found by using a protractor.

Example. Figure 10-8 represents a Mercator chart. Approximately how many miles are represented by lines BC, BA, and AC?

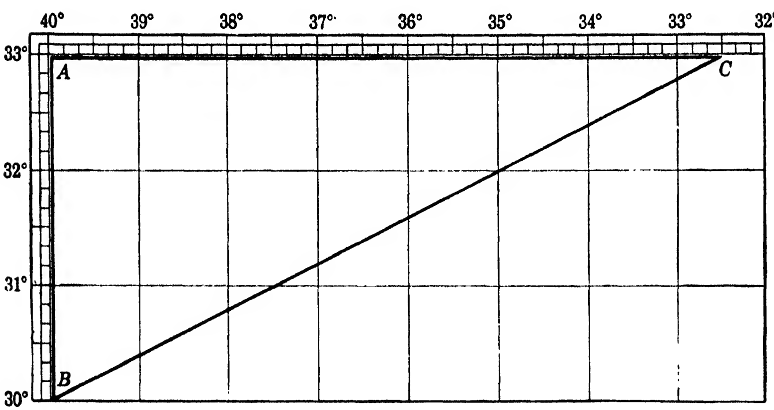


FIG. 10-8.

Solution. Using dividers, we lay off BC along the longitude scale and read 507′. The mid-latitude is 31.5°. Hence, in accordance with (4), BC represents the length d given by

$$d = 507 \cos 31.5° = 432 \text{ miles.}$$

The student should also find this result by reading the latitudes of B and C from the chart, measuring the course angle with a protractor, and applying formula (3).

Similarly $BA = 210'$. Hence

$$l = 210 \cos 31.5° = \textbf{180 miles.}$$

Observe that it is the difference in latitude for the track BC. This could have been found by observing that BA represents the three degrees of latitude from $30'$ to $33°$ on the left of the chart. Hence it represents $3 \times 60 = 180$ miles.

Likewise, $AC = 450'$, and AC lies in Lat. $33°$. Hence in accordance with (4) it represents the length p given by

$$p = 450' \cos 33° = \textbf{378 miles.}$$

Observe that this is the departure for track BC.

EXERCISES 10-3

1. In Fig. 10-9 find approximately how many miles are represented by DE, EF, and FD.

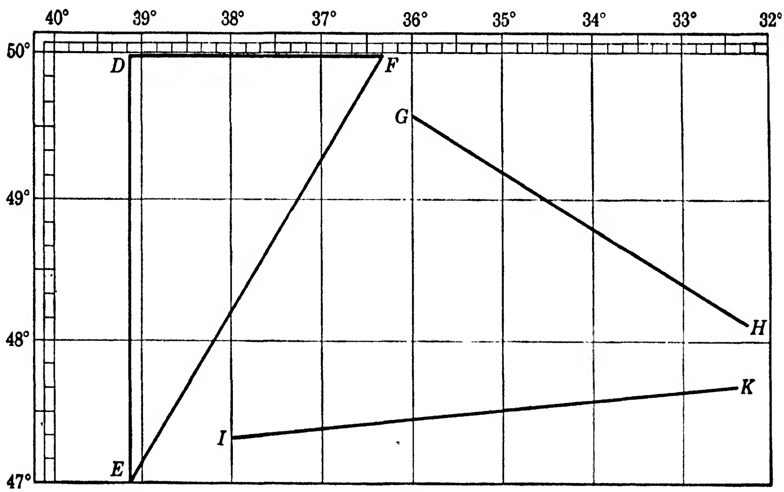
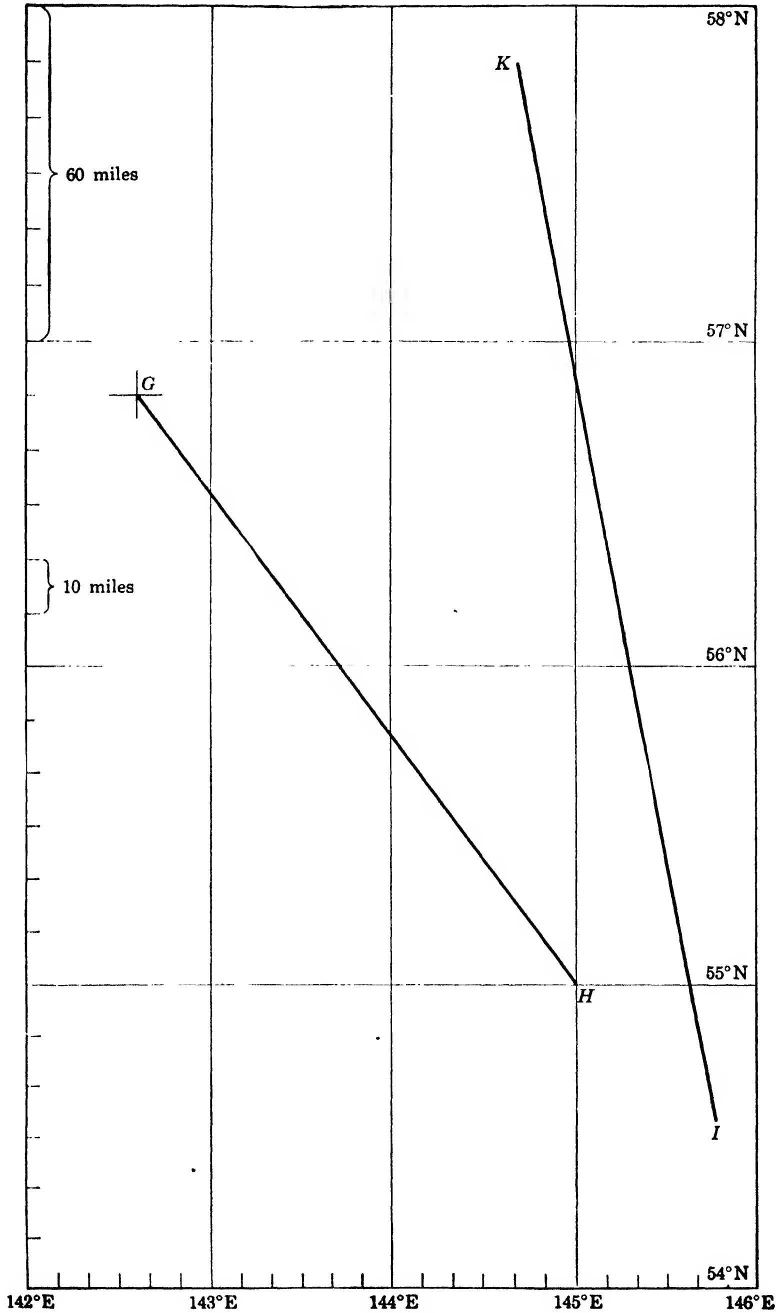


FIG. 10-9.

2. Read from the chart of Fig. 10-9 the latitude and longitude of each point lettered.

3. Using formula (6) find the rhumb-line distance represented by each of the following lines in Fig. 10-9: (*a*) GH, (*b*) IK. Also find these distances by reading the latitudes of the end points measuring the angle C and using (5).


58° N
57° N
56° N
55° N
54° N
60 miles
10 miles
K
G
H
I
142° E
143° E
144° E
145° E
146° E
Fig. 10-10.

4. Plot on Fig. 10-9 point M_1, $L = 49°20'$ N., $\lambda = 38°$ W., and point M_2, $L = 47°30'$ N., $\lambda = 32°30'$ W. Draw a line connecting these points and measure the angle (course angle) this line makes with a meridian. Obtain the difference in latitude between the ends of the line and use formula (5) to find the number of miles it represents.

5. If a ship sails from G to H (see Fig. 10-10), find the difference in latitude and the difference in longitude (*a*) by reading these quantities directly from the figure, (*b*) by using formulas (6) and (5). *Hint.* Measure d_m and C, find d from (6), and $L_2 - L_1$ from (5).

6. In exercise 5 replace G by K and H by I and then solve the problem.

7. From a point M_1 in Lat. 47°30′ N., Long. 39°40′ W., draw a line at an angle of 50° with the meridians and running upward and toward the right a distance of 2 in. At the upper end of this line segment make a dot and mark it M_2. Find the latitude and longitude of M_2 (*a*) by reading these quantities from the chart, (*b*) by using formulas (4) and (5). Use Fig. 10-9.

8. A ship steams from a point in Lat. 47°30′ N., Long. 36°10′ W., to a second point in Lat. 49°10′ N., Long. 33°50′ W. Using Fig. 10-9, find the rhumb-line distance between the two points and the rhumb-line course angle. (Measure the course angle with a protractor.)

9. A ship steams on a rhumb-line course of 70° for a distance of 45 miles from a point in Lat. 30°20′ N., Long. 30°20′ W., to a second point. Find the latitude and longitude of the second point.

10. With each of the following trips the rhumb-line distance is tabulated. W represents westward sailing; E represents eastward sailing. Using (7) find, in each case, the course C_n.

	Distance
(*a*) San Francisco, $L = 37°48'$N., to Honolulu, $L = 21°18'$N.	W 2100 mi.
(*b*) Honolulu, $L = 21°18'$N., to Manila, $L = 14°36'$N.	W 2160 mi.
(*c*) Manila, $L = 14°36'$N., to Tokyo, $L = 35°39'$N.	E 1620 mi.
(*d*) Tokyo, $L = 35°39'$N., to Singapore, $L = 1°18'$N.	W 2880 mi.

11. Seattle is situated in Lat. 47°36′ N., Long. 122°20′ W.; Bangor, Me., in Lat. 44°48′ N., Long. 68°47′ W.; and Pensacola in Lat. 30°21′ N., Long. 87°19′ W. The meridional parts for latitudes 47°36′, 44°48′, 30°21′ are, respectively, 3238.5, 2996.5, and 1900.8. Find the course and distance for a rhumb-line flight (*a*) from Seattle to Bangor, (*b*) from Bangor to Pensacola, (*c*) from Pensacola to Seattle.

12. Using the chart of Fig. 10-11, find the rhumb-line course and distance from Malta to Alexandria. *Hint.* Use formula (5).

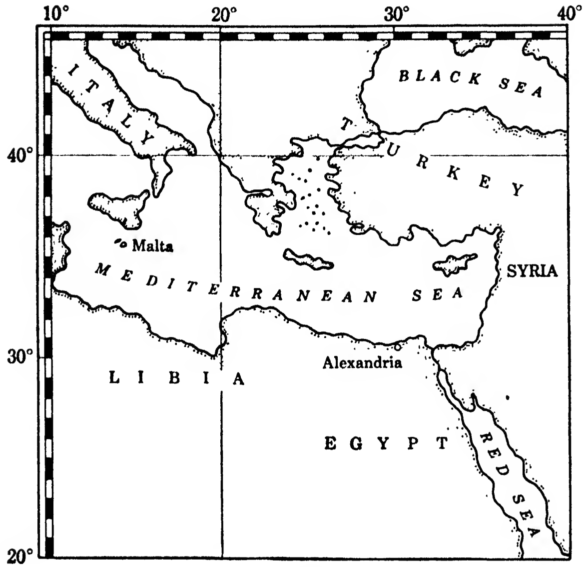


FIG. 10-11.

13. Using the chart of Fig. 10-12, find the course and distance by rhumb line for each of the following trips:

(*a*) Honolulu to Manila. (*b*) Honolulu to Batavia.
(*c*) Honolulu to Tokyo. (*d*) Honolulu to Guam.
(*e*) Honolulu to Wake Island. (*f*) Guam to Tokyo.
(*g*) Wake Island to Tokyo. (*h*) Guam to Manila.

14. With each of the following trips the course C_n is tabulated. Using (7) find, in each case, the rhumb-line distance:

	Course
(*a*) Singapore, $L = 1°18'$ N., to Darwin, $L = 12°23'$ S.	117°5′
(*b*) New York, $L = 40°42'$ N., to Liverpool, $L = 53°27'$ N.	75°10′
(*c*) Dakar, $L = 14°41'$ N., to Natal, Brazil, $L = 5°47'$ S.	221°

10-7. Problems involving the solution of right spherical triangles. A number of elementary problems are solved by means of right spherical triangles. The following examples will serve as illustrations.

Example 1. An airplane leaves Seattle, $L = 47°36'$ N., $\lambda = 122°20'$ W., on course 300° and flies along a great-circle

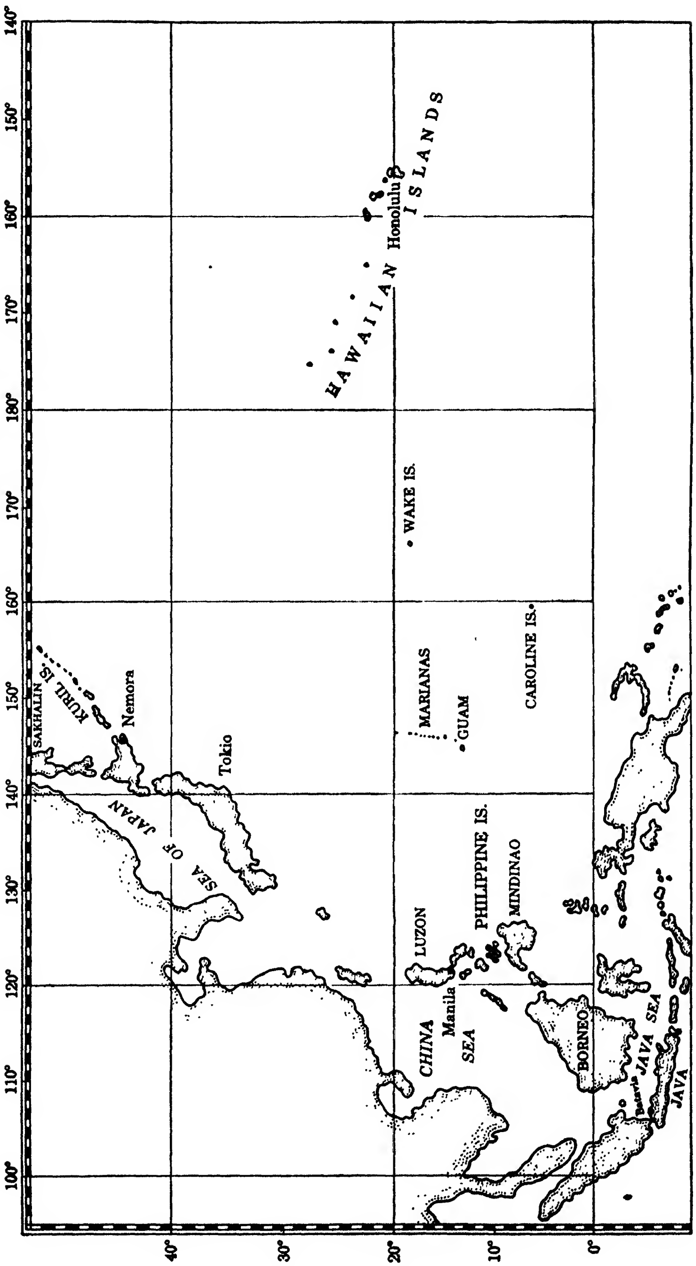

140°
150°
160°
170°
180°
170°
160°
150°
140°
130°
120°
110°
100°
40°
30°
20°
10°
0°
HAWAIIAN ISLANDS
Honolulu
WAKE IS.
CAROLINE IS.
MARIANAS
GUAM
SAKHALIN
KURIL IS.
Nemora
Tokio
SEA OF JAPAN
PHILIPPINE IS.
MINDINAO
LUZON
Manila
CHINA
SEA
BORNEO
JAVA SEA
JAVA
Batavia

FIG. 10-12.

track. Find the latitude and longitude of the northernmost point on this track and its distance from Seattle.

Solution. In Fig. 10-13, great circle SV represents the track with its northernmost point on vertex V. Evidently, great circle P_nV is perpendicular to great circle SV, and P_nSV is a right triangle with the right angle at V. The colatitude (90° − Lat.) P_nS of Seattle and the course angles are known. By applying Napier's rules, we obtain

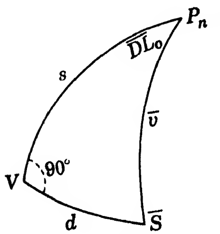


FIG. 10-13.

$$\cot = \cos v \tan s, \qquad (8)$$
$$\tan d = \tan v \cos S, \qquad (9)$$
$$\sin s = \sin v \sin S, \qquad (10)$$
$$\sin s = \tan d \cot DLo. \quad (\textit{Check}) \qquad (11)$$

The following form contains the solution of the triangle:

	(8)(11)	(9)	(10)
$v = 90° - L = 42°24'$	l cos 9.8683	l tan 9.9605	l sin 9.8289
$S = 60°$	l tan 0.2386	l cos 9.6990	l sin 9.9375
$DLo = 38°1'$	l cot 0.1069		
$d = 24°32'$	l tan 9.6595	l tan 9.6595	
$s = 35°44'$	l sin 9.7664		l sin 9.7664

The latitude of the vertex is $90° - s =$ **54°16′ N.**, its longitude is (Long. $S + DLo$) = 122°20′ W. + 38°1′ W. = **160°21′ W.**, and the distance s in nautical miles is $d = (24)(60) + 32 =$ **1472 miles.**

Example 2. Find the shortest distance from New York, $L = 40°42'$ N., $\lambda = 73°59'$ W., to the meridian of Greenwich, England.

Solution. In Fig. 10-14, Y represents New York, P_n the north pole, and M a point on the zero meridian. Evidently,

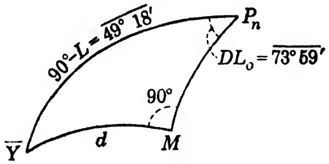


FIG. 10-14.

$$YP_n = 90° - \text{Lat.}$$

or 49°18′, and $DLo = 73°59'$. Using Napier's rules, we get $\sin d = \sin 49°18' \sin 73°59'$. Computing d from this formula, we obtain

$$d = 46°30' = (60)(46) + 30 = \textbf{2790 miles.}$$

EXERCISES 10-4

1. An airplane following a great-circle track flies from New York, $L = 40°42'$ N., $\lambda = 73°55'$ W., to Moscow, $L = 55°50'$ N., $\lambda = 37°35'$ E. Find the latitude and the longitude of the vertex of the track.

2. In each case find the latitude and the longitude of the northernmost vertex of the following great-circle tracks taken by an airplane:

(*a*) From Honolulu, $L = 21°25'$ N., $\lambda = 157°55'$ W. to New York, $L = 40°42'$ N., $\lambda = 73°55'$ W.

(*b*) From Honolulu to Manila, $L = 14°40'$ N., $\lambda = 121°0'$ E.

(*c*) From Manila to Chungking, $L = 29°40'$ N., $\lambda = 106°5'$ E.

3. In each case find the southern vertex of the following great-circle tracks flown by an airplane:

(*a*) From Fairbanks, $L = 64°47'$ N., $\lambda = 147°46'$ W., to New York.

(*b*) From Fairbanks to Chungking.

4. In each case find the northern vertex of the following great-circle tracks taken by an airplane:

(*a*) From Oslo, Norway, $L = 59°55'$ N., $\lambda = 10°43'$ E., to Seattle, $L = 47°40'$ N., $\lambda = 122°15'$ W.

(*b*) From Oslo to Washington, D.C., $L = 38°52'$ N., $\lambda = 77°0'$ W.

(*c*) From Oslo to Chicago, $L = 41°50'$ N., $\lambda = 87°37'$ W.

5. For the track of Exercise 3(*a*), find

(*a*) The latitude and the longitude of the nearest approach to San Francisco, $L = 37°43'$ N., $\lambda = 122°25'$ W.

(*b*) The length of the shortest distance from San Francisco to the place of nearest approach.

6. Find the shortest distance from New York to the meridian of Bermuda, $L = 32°15'$ N., $\lambda = 64°50'$ W.

7. A ship sails initially due east along a great-circle track from Norfolk, Va., $L = 36°51'$ N., $\lambda = 76°16'$ W., for 1000 miles. Find the latitude and the longitude of the position reached.

8. (*a*) If the difference of longitude of two places A and B on the earth is 50° and their latitudes are 30°, find the distance AB measured on the equal latitude circle.

(*b*) What is the distance AB measured on a great circle? The radius of the earth is approximately 3960 land miles.

9. Two points A and B are the ends of a 500-land-mile arc of a small circle in Lat. 36° N. Find the difference in their longitudes. If A_1 and B_1 are both in Lat. 36° N. and the arc of a great circle connecting them is 500 land miles long, what is the difference in their longitudes? Assume the radius of the earth to be 3960 land miles.

10. The initial course of a certain ship sailing from New York is due east. After she has sailed 600 nautical miles on a great circle, find her latitude, longitude, and course.

11. Find the latitude and distance from New York of the ship in Exercise 10 when her longitude is 15°25′ W.

12. A ship departs from A in Long. 22° W. and sails 218 miles due west to B in Long. 27°12′ W. Along what parallel did she sail?

13. At what rate per hour is the Greenwich Observatory in Lat. 51°28.5′ N. being carried around the earth's axis? *Hint.* The earth rotates through 360° in 24 hr. or 15° of longitude per hour, which is at the rate of 900′ per hour at the equator.

CHAPTER 11

THE OBLIQUE SPHERICAL TRIANGLE

11-1. The six cases. When three parts of a spherical triangle are given, the other three parts can be computed. Accordingly, a classification of spherical triangles is made on the basis of given parts. Six cases are referred to as follows:

I. Given two sides and the included angle.

II. Given two angles and the included side.

III. Given the three sides.

IV. Given the three angles.

V. Given two sides and the angle opposite one of them.

VI. Given two angles and the side opposite one of them.

For the purposes of solution, there are, in a sense, only three cases. If a method of solution for Case I is known, this same method may be applied to solve the polar triangle of a triangle classified under Case II. The solution of a quadrantal triangle in Art. 9-12 by the method of solving a right spherical triangle illustrates this process. Similarly, the formulas used to solve a triangle under Case III may be used to solve the polar of a triangle classified under Case IV. Also, the same formulas may be used to solve a triangle coming under Case V and the polar of a triangle classified under Case VI.

Case I is by far the most important for navigation. A method of solving this case by means of the right spherical triangle was treated in Art. 9-13.

To deduce other methods of solving spherical triangles, we shall develop general formulas analogous to those used with plane triangles.

11-2. The law of sines. The law of sines for spherical triangles may be stated as follows:

The sines of the sides of a spherical triangle are proportional to the sines of the angles opposite, or in symbols

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{1}$$

In Fig. 11-1 let a, b, c represent the sides of a spherical triangle and let A, B, C represent the opposite angles. Draw an arc $CD(= h)$ of a great circle through the vertex C perpendicular to the side c, or the side c produced, to form the right spherical

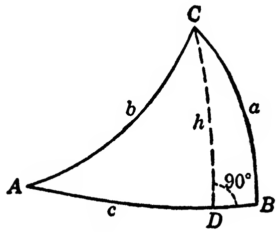


FIG. 11-1a.

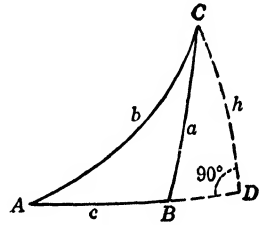


FIG. 11-1b.

triangles ACD and BCD. Apply Napier's rules to these right triangles to obtain

$$\sin h = \sin b \sin A, \qquad \sin h = \sin a \sin B.$$

Equating these two values of $\sin h$, we get

$$\sin a \sin B = \sin b \sin A,$$

and, dividing by $\sin A \sin B$,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}. \tag{2}$$

In like manner, by drawing an arc from A perpendicular to CB and arguing as above, we can show that

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{3}$$

Equations (2) and (3) are together equivalent to (1). The law of sines may be used in the solution of a spherical triangle when a side and the angle opposite are included among the given parts.

When a part of a spherical triangle is found by means of the law of sines, there is often some difficulty in determining whether the part found is of the first quadrant or of the second quadrant; for $\sin A = \sin (180° - A)$. Other formulas may be used in many cases. However, the following theorems from solid geometry will often enable the computer to determine the quadrant.

The order of magnitude of the sides of a spherical triangle is the same as the order of magnitude of the respective opposite

angles; or, in symbols, if

$$a < b < c, \qquad \text{then} \qquad A < B < C.$$

The sum of two sides of a spherical triangle is greater than the third side.

EXERCISES 11-1

1. Figure 11-2 represents the spherical triangle ABC with its associated trihedral angle O, the face angles of which are a, b, c. AF is the intersection of two planes, one perpendicular to OB, the other perpendicular to OC. Point F is in plane OCB. Taking $OA = 1$ unit, express the values of all straight-line segments of the figure in terms of a, b, c, B, and C. Derive the law of sines from the result.

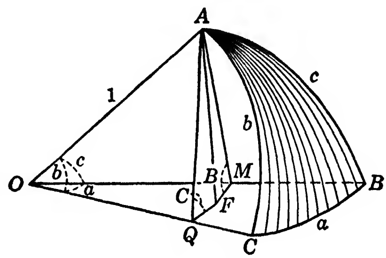


FIG. 11-2.

2. Check the following data by using the law of sines:

(a) $A = 108°40'$, $B = 134°20'$, $C = 70°18'$, $a = 145°36'$, $b = 154°45'$, $c = 34°9'$.
(b) $A = 47°21'$, $B = 22°20'$, $C = 146°40'$, $a = 117°9'$, $b = 27°22'$, $c = 138°20'$.
(c) $A = 110°10'$, $B = 133°18'$, $C = 70°16'$, $a = 147°6'$, $b = 155°5'$, $c = 32°59'$.

3. Use the law of sines to find the missing parts of the following right spherical triangles:

(a) $a = 58°8'$, $b = 32°49'$, $B = 37°12'$, $c = 63°40'$.
(b) $a = 36°14'$, $A = 49°29'$, $b = 38°45'$, $c = 51°1'$.

4. Use the law of sines to find the missing part of each of the following spherical triangles:

(a) $A = 130°5'$, $B = 32°26'$, $C = 36°45'$, $c = 51°6'$, $a = 84°14'$.
(b) $A = 70°$, $C = 94°48'$, $c = 116°$, $a = 57°56'$, $b = 137°20'$.

5. Solve the polar triangles of the triangles of Exercise 3.

11-3. The law of cosines for sides. The cosine of any side of a spherical triangle is equal to the product of the cosines of the two other sides increased by the product of the sines of the two other sides and the cosine of the angle included between them, or, in symbols,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{4}$$

The following proof is analogous to the one given for the law of cosines in plane trigonometry.

In Fig. 11-1 let arc $AD = \varphi$. Then arc $BD = c - \varphi$. Write these values on the triangle of Fig. 1(*a*), and place bars over a, b, A, and B in preparation for using Napier's rules. The result is Fig. 11-3.

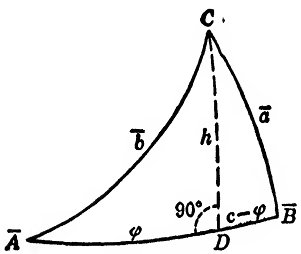


FIG. 11-3.

Now apply Napier's rules to triangles ACD and BCD to obtain

$$\cos a = \cos h \cos (c - \varphi), \tag{5}$$
$$\cos b = \cos h \cos \varphi. \tag{6}$$

Divide (5) by (6), member by member, and transform slightly to get

$$\frac{\cos a}{\cos b} = \frac{\cos h \cos (c - \varphi)}{\cos h \cos \varphi} = \frac{\cos c \cos \varphi + \sin c \sin \varphi}{\cos \varphi}, \tag{7}$$

or, simplifying further,

$$\cos a = \cos b(\cos c + \sin c \tan \varphi). \tag{8}$$

Again apply Napier's rules, using parts b, A, φ of triangle ACD to obtain

$$\cos A = \cot b \tan \varphi,$$

or

$$\tan \varphi = \cos A \tan b. \tag{9}$$

Replace $\tan \varphi$ in (8) by its value from (9) to get

$$\cos a = \cos b(\cos c + \sin c \cos A \tan b), \tag{10}$$

or, simplifying the right-hand member,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \tag{11}$$

Similarly, we may obtain

$$\cos b = \cos a \cos c + \sin a \sin c \cos B, \tag{12}$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{13}$$

An argument differing slightly from the one just used shows that (1) holds for a triangle shaped like the triangle of Fig. 11-1(b).

The law of cosines applies to the solution of a spherical triangle when two sides and the included angle are given, and also when the three sides are given. Although it is not adapted to logarithmic computation, logarithms may be used as in the following examples.

Example 1. Find c in the spherical triangle in which $a = 64°24'$, $b = 43°20'$, and $C = 58°40'$.

Solution. Substituting in the formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos C,$$

we obtain

$$\cos c = \cos 64°24' \cos 43°20' + \sin 64°24' \sin 43°20' \cos 58°40'.$$

Here it will be necessary to compute each product by the use of natural functions, add the results, and then find the value of c from the table of natural cosines; or find the logarithm of the cosine of c and then find c from the table giving the logarithms of the cosines. The following solution shows how logarithms may be used in finding the products in the right-hand member.

	($\cos a \cos b$)	($\sin a \sin b \cos C$)
$a = 64°24'$	l cos 9.6356	l sin 9.9551
$b = 43°20'$	l cos 9.8618	l sin 9.8365
$C = 58°40'$		l cos 9.7160
0.3144	log 9.4974	
0.3218		log 9.5076
0.6362		

$$\therefore c = \cos^{-1}(0.6362) = \mathbf{50°30'}.$$

Example 2. Find A in Example 1, given $a = 64°24'$, $b = 43°20'$, and $c = 50°30'$.

Solution. From the formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

we obtain
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Substituting the given values and using logarithms to get the products in the right~~-hand mem~~ber,

$$\begin{array}{lccc}
 & (\cos a) & (\cos b \cos c) & (\sin b \sin c) \\
a = 64°24' & 0.4321 & & \\
b = 43°20' & & \text{l cos } 9.8618 & \text{l sin } 9.8365 \\
c = 50°30' & & \text{l cos } 9.8035 & \text{l sin } 9.8874 \\
 & 0.4627 & \log 9.6653 & \\
 & (-)0.0306 \div 0.5295 & & \log 9.7239
\end{array}$$

$$A = \cos^{-1}(-0.0578) = \mathbf{93°19'}.$$

EXERCISES 11-2

1. Use the law of cosines to find a in each of the following spherical triangles:

(*a*) $b = 60°$, $c = 30°$, $A = 45°$.
(*b*) $b = 45°$, $c = 30°$, $A = 120°$.
(*c*) $b = 45°$, $c = 60°$, $A = 150°$.

2. Solve the following triangles for the part indicated:

(*a*) $b = 93°$, $c = 46°4'$, $A = 71°6'$, $a = ?$
(*b*) $c = 58°6'$, $a = 22°8'$, $B = 112°0'$, $b = ?$
(*c*) $a = 127°46'$, $b = 65°32'$, $C = 94°38'$, $c = ?$
(*d*) $c = 58°49'$, $a = 34°47'$, $B = 36°22'$, $b = ?$
(*e*) $b = 95°26'$, $a = 43°24'$, $C = 69°36'$, $c = ?$
(*f*) $c = 123°31'$, $a = 84°47'$, $B = 125°4'$, $b = ?$
(*g*) $c = 39°0'$, $a = 49°0'$, $b = 62°0'$, $A = ?$
(*h*) $b = 61°0'$, $c = 43°0'$, $a = 68°0'$, $B = ?$
(*i*) $b = 147°40'$, $c = 72°10'$, $a = 121°36'$, $C = ?$
(*j*) $c = 81°27'$, $a = 38°50'$, $b = 92°43'$, $C = ?$
(*k*) $c = 115°36'$, $a = 56°2'$, $b = 96°52'$, $A = ?$
(*l*) $b = 107°54'$, $c = 68°4'$, $a = 64°10'$, $B = ?$

11-4. The law of cosines for angles. When the three angles of a spherical triangle are given, the law of cosines can be applied to its polar triangle in which the values of the sides are also known. The angles of the polar triangle can thus be obtained, and from these the values of the sides of the given triangle will be known.

Another method is to use the law of cosines for angles, which is developed as follows:

Applying (11) to the polar triangle (see Art. 9-11) of ABC, we obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'. \tag{14}$$

Using equation (11) of Art. 9-11 to replace a', b', c', and A' of (14) by $180° - A$, $180° - B$, $180° - C$, and $180° - a$, respectively, we obtain

$$\cos (180° - A) = \cos (180° - B) \cos (180° - C) + \sin (180° - B) \sin (180° - C) \cos (180° - a),$$

or

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a,$$

or

$$\boldsymbol{\cos A = -\cos B \cos C + \sin B \sin C \cos a.} \tag{15}$$

Similarly, we obtain from (12) and (13)

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b, \tag{16}$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{17}$$

Evidently this process of applying known formulas to the polar triangle of a given one is very important. It furnishes a method of deriving from every equation applying to a general spherical triangle another equation that may be called the *dual* of the first one. The role played by the sides in the given equation is played by the angles in the dual equation, and the role played by the angles in the given equation is played by the sides in the other. A similar statement applies to theorems relating to a spherical triangle. This principle of duality will come to our attention again and again in the discussion that follows.

Example. In a certain spherical triangle, $A = 60°$, $B = 60°$, and $c = 60°$. Find C.

Solution. Substituting 60° for each of the letters A, B, and c in (17), we obtain

$$\cos C = -\cos 60° \cos 60° + \sin 60° \sin 60° \cos 60°$$
$$= -\tfrac{1}{4} + \tfrac{3}{8} = \tfrac{1}{8}.$$

Hence

$$C = \cos^{-1} \tfrac{1}{8} = \mathbf{82°49'}.$$

EXERCISES 11-3

1. Use the law of cosines for angles to find A for each of the following triangles:

(*a*) $B = 120°$,	(*b*) $B = 135°$,	(*c*) $a = 30°$,
$C = 150°$,	$C = 120°$,	$B = 150°$,
$a = 135°$.	$a = 30°$.	$C = 135°$.

2. Solve the following triangles for the part indicated:

(*a*) $a = 78°46'$,	(*b*) $b = 68°20'$,	(*c*) $c = 108°10'$,
$B = 105°36'$,	$A = 57°64'$,	$A = 123°59'$,
$C = 44°0'$,	$C = 22°6'$,	$B = 72°43'$,
$A = ?$	$B = ?$	$C = ?$
(*d*) $C = 45°16'$,	(*e*) $B = 70°32'$,	(*f*) $C = 72°46'$,
$A = 58°24'$,	$C = 47°20'$,	$A = 37°16'$,
$B = 94°52'$,	$A = 88°4'$,	$B = 105°16'$,
$a = ?$	$b = ?$	$c = ?$

3. Derive the law of sines algebraically from the law of cosines.

Hint. Solve (11) for $\cos A$, form $\sin^2 A$, and reduce the numerator to a form involving cosines only. Then show that $\sin^2 A/\sin^2 a$ is symmetrical in a, b, c.

4. In Fig. 11-4, ABC represents a spherical triangle with its associated trihedral angle O. BLM is a plane through B perpendicular to OB, intersecting OA produced, in M and OC produced, in L. Taking $OB = 1$ unit, express the values of the line segments OL, OM, BL, BM in terms of a, b, c, then apply the law of cosines of plane trigonometry to the triangles BLM, and OLM, and equate two values of $\overline{LM}^2$ to obtain after slight transformation

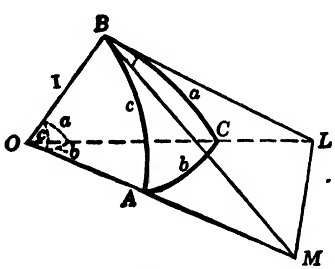


FIG. 11-4.

$$\cos b = \cos a \cos c + \sin a \sin c \cos B.$$

5. In each of the triangles of Exercise 1 complete the solution by means of the law of sines.

6. Using the law of cosines, prove that in a spherical triangle having three sides of the second quadrant the angles opposite are of the second quadrant.

7. Replace C by $90°$ in (1), (13), (15), and (17), and then obtain the resulting formulas by applying Napier's rules to the parts of a right spherical triangle.

11-5. The half-angle formulas. These are derived from the law of cosines and are better adapted to the use of logarithms. Solving (11) for $\cos A$,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{18}$$

Equating 1 minus the left-hand member to 1 minus the right-hand member and simplifying slightly, we get

$$1 - \cos A = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

or, replacing $\sin b \sin c + \cos b \cos c$ by $\cos (b - c)$,

$$1 - \cos A = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

Now, replacing $1 - \cos A$ by $2 \sin^2 \frac{1}{2}A$ and changing the right-hand member by using (36) of Art. 6-6 and the fact that $\sin (-\theta) = -\sin \theta$, we get

$$2 \sin^2 \tfrac{1}{2}A = \frac{2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}. \tag{19}$$

Denote half the sum of the sides by s and write

$$s = \tfrac{1}{2}(a + b + c). \tag{20}$$

Subtracting in succession a, b and c from both members of (20), we obtain

$$\left.\begin{array}{c} s - a = \frac{1}{2}(-a + b + c), \qquad s - b = \frac{1}{2}(a - b + c), \\ s - c = \frac{1}{2}(a + b - c). \end{array}\right\} \tag{21}$$

Substituting from (21) in (19) and taking the square root of both members, we obtain

$$\sin \tfrac{1}{2}A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}. \tag{22}$$

Considerations of symmetry show that

$$\sin \tfrac{1}{2}B = \sqrt{\frac{\sin (s-a) \sin (s-c)}{\sin a \sin c}}, \tag{23}$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}}. \tag{24}$$

Similarly, proceeding as above, we obtain

$$\begin{aligned} 1 + \cos A &= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \\ &= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c}, \\ &= \frac{\cos a - \cos (b + c)}{\sin b \sin c}, \end{aligned}$$

$$1 + \cos A = \frac{2 \sin \tfrac{1}{2}(a + b + c) \sin \tfrac{1}{2}(-a + b + c)}{\sin b \sin c}. \tag{25}$$

Replacing in (25) $1 + \cos A$ by $2 \cos^2 \tfrac{1}{2}A$, using (20) and (21) and extracting the square root of both members, we get

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}}. \tag{26}$$

Considerations of symmetry show that

$$\cos \tfrac{1}{2}B = \sqrt{\frac{\sin s \sin (s-b)}{\sin a \sin c}}, \tag{27}$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}. \tag{28}$$

Dividing (22) by (26), member by member, and replacing $\sin \tfrac{1}{2}A \div \cos \tfrac{1}{2}A$ by $\tan \tfrac{1}{2}A$, we obtain

$$\tan \tfrac{1}{2}A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}}. \tag{29}$$

Multiplying numerator and denominator under the radical by $\sin (s - a)$ and removing $1/\sin^2 (s - a)$ from the radical, we have

$$\tan \tfrac{1}{2}A = \frac{1}{\sin (s-a)} \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}, \tag{30}$$

or

$$\tan \tfrac{1}{2}A = \frac{r}{\sin (s-a)}, \tag{31}$$

where

$$r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \tag{32}$$

Similarly,

$$\tan \tfrac{1}{2}B = \frac{r}{\sin(s-b)}, \tag{33}$$

$$\tan \tfrac{1}{2}C = \frac{r}{\sin(s-c)}. \tag{34}$$

11-6. Solution of a triangle by the use of the half-angle formulas. Evidently formulas (31), (33), and (34) are adapted to solve a spherical triangle when three sides are given. To solve a spherical triangle when the three angles are given, we find the sides of the polar triangle by subtracting each of the given angles from 180° and then applying equations (31), (33), and (34) to find the angles of the polar triangle; subtraction of each of these angles from 180° gives the sides of the original triangle.

Example. Find A, B, and C for a spherical triangle in which $a = 64°24'$, $b = 43°20'$, and $c = 50°30'$.

Solution. $s = \frac{1}{2}(a + b + c) = 79°7'$. The solution by means of the half-angle formulas (32), (31), (33), (34) and the check by the law of sines follow.

$s - a = 14°43'$	l sin 9.4049	col sin 0.5951		
$s - b = 35°47'$	l sin 9.7670		col sin 0.2330	
$s - c = 28°37'$	l sin 9.6803			col sin 0.3197
$s = 79°7'$	col sin 0.0079			
2	log 8.8601			
r	log 9.4301	log 9.4301	log 9.4301	log 9.4301
$\frac{1}{2}A = 46°39.6'$		tan 0.0252		
$A =$ **93°19'**				
$\frac{1}{2}B = 24°43'$			tan 9.6631	
$B =$ **49°26'**				
$\frac{1}{2}C = 29°20.3'$				tan 9.7498
$C =$ **58°41'**				

Check.	a l sin 9.9551	b l sin 9.8365	c l sin 9.8874
	A l sin 9.9993	B l sin 9.8806	C l sin 9.9316
	9.9558	9.9559	9.9558

EXERCISES 11-4

1. Solve the following spherical triangles:

(a) $a = 30°$, $b = 45°$, $c = 60°$.
(b) $a = 30°$, $b = 60°$, $c = 60°$.
(c) $a = 150°$, $b = 120°$, $c = 60°$.
(d) $a = 110°$, $b = 32°$, $c = 96°$.
(e) $a = 108°14'$, $b = 75°29'$, $c = 56°37'$.
(f) $a = 78°15'$, $b = 101°20'$, $c = 112°38'$.

2. Solve the following spherical triangles by using their polar triangles:

(a) $A = 60°$, $B = 135°$, $C = 60°$.
(b) $A = 150°$, $B = 120°$, $C = 135°$.
(c) $A = 80°$, $B = 110°$, $C = 130°$.
(d) $A = 59°55'$, $B = 85°37'$, $C = 59°55'$.
(e) $A = 89°6'$, $B = 54°32'$, $C = 102°14'$.
(f) $A = 172°18'$, $B = 8°28'$, $C = 4°24'$.

3. Derive the following equations from (22) to (34):

$$\frac{\cos \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C} = \frac{\sin s}{\sin c},$$
$$\frac{\cos \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}C} = \frac{\sin (s - a)}{\sin c},$$
$$\frac{\sin \frac{1}{2}A \cos \frac{1}{2}B}{\cos \frac{1}{2}C} = \frac{\sin (s - b)}{\sin c},$$
$$\frac{\sin \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}C} = \frac{\sin (s - c)}{\sin c}.$$

4. Prove that the following relation holds true for a right spherica triangle:

$$\tan^2 \tfrac{1}{2}A = \sin (c - b) \csc (c + b).$$

5. Write $\sigma = \dfrac{A + B + C}{2}$, and use equations (11) of Art. 9-11 to derive

$$s' = \frac{a' + b' + c'}{2} = 270° - \frac{A + B + C}{2} = 270° - \sigma,$$
$$s' - a' = 90° - (\sigma - A), \qquad s' - b' = 90° - (\sigma - B),$$
$$s' - c' = 90° - (\sigma - C).$$

Then apply equations (22), (26), and (29), to the polar triangle to obtain

$$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos(\sigma - B)\cos(\sigma - C)}{\sin B \sin C}},$$

$$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos \sigma \cos(\sigma - A)}{\sin B \sin C}},$$

$$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos \sigma \cos(\sigma - A)}{\cos(\sigma - B)\cos(\sigma - C)}}.$$

11-7. Napier's analogies. This article is devoted to deriving formulas that may be used to solve triangles for which the given parts are two sides and the included angle or two angles and the included side. Substituting $\frac{1}{2}A$ for A and $\frac{1}{2}B$ for B in (7) and (10) of Art. 6-2, we get

$$\sin \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B, \tag{35}$$

$$\sin \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B - \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B. \tag{36}$$

Dividing (36) by (35), member by member, we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B}{\sin \frac{1}{2}A \cos \frac{1}{2}B + \cos \frac{1}{2}A \sin \frac{1}{2}B}. \tag{37}$$

Or, dividing both numerator and denominator of the right-hand member of (37) by $\sin \frac{1}{2}A \sin \frac{1}{2}B$,

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{\cot \frac{1}{2}A - \cot \frac{1}{2}B}{\cot \frac{1}{2}A + \cot \frac{1}{2}B}. \tag{38}$$

From (31) and (33) we find $\cot \frac{1}{2}A = \dfrac{\sin(s - a)}{r}$ and

$$\cot \tfrac{1}{2}B = \frac{\sin(s - b)}{r}.$$

Substituting these values in (38) and canceling r, we obtain

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{\sin(s - a) - \sin(s - b)}{\sin(s - a) + \sin(s - b)}. \tag{39}$$

Using (34) and (33) of Art. 6-6 to transform the right-hand member of (39), we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = -\frac{2 \cos \frac{1}{2}(2s - a - b) \sin \frac{1}{2}(b - a)}{2 \sin \frac{1}{2}(2s - a - b) \cos \frac{1}{2}(b - a)}. \tag{40}$$

Replacing $(2s - a - b)$ by c in (40) and simplifying slightly, we get

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c}. \tag{41}$$

Again, using (11) and (8) of Art. 6-2 with $A = \frac{1}{2}A$ and $B = \frac{1}{2}B$, we get

$$\cos \tfrac{1}{2}(A - B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B + \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B, \tag{42}$$
$$\cos \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B - \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B. \tag{43}$$

Dividing (42) by (43), member by member, then dividing numerator and denominator of the right-hand member of the resulting equation by $\sin \frac{1}{2}A \sin \frac{1}{2}B$ and finally replacing $\cot \frac{1}{2}A$ by $\dfrac{\sin (s - a)}{r}$ and $\cot \frac{1}{2}B$ by $\dfrac{\sin (s - b)}{r}$, we have

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\dfrac{\sin (s - a) \sin (s - b)}{r^2} + 1}{\dfrac{\sin (s - a) \sin (s - b)}{r^2} - 1}. \tag{44}$$

Replacing r^2 by its value from (32) and simplifying slightly, we obtain

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\sin s + \sin (s - c)}{\sin s - \sin (s - c)}. \tag{45}$$

Treating the right-hand member of this equation in a manner similar to that employed in transforming (39), we get

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c}. \tag{46}$$

Applying (41) and (46) to the polar triangle, we obtain

$$\frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A - B)}{\cot \frac{1}{2}C}, \tag{47}$$

$$\frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} = \frac{\tan \frac{1}{2}(A + B)}{\cot \frac{1}{2}C}. \tag{48}$$

The formulas (41), (46), (47), and (48) are known as Napier's analogies. These formulas are analogous to the law of tangents in plane trigonometry.

EXERCISES 11-5

1. Apply (41) and (46) to the polar triangle, then proceed in a manner analogous to that pursued in this article and obtain formulas (47) and (48).

2. Use formulas (41), (46), (47), and (48) to prove the following formulas known as Gauss's equations or Delambre's analogies:

$$\sin \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\sin \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}c} \cos \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}c} \sin \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}c} \sin \tfrac{1}{2}C.$$

3. From formula (46), show that in any spherical triangle one-half the sum of two angles is in the same quadrant as one-half the sum of the opposite sides; that is, $\frac{1}{2}(a + b)$ and $\frac{1}{2}(A + B)$ are in the same quadrant.

4. (*a*) Divide $\sin \frac{1}{2}(A - B) = \sin \frac{1}{2}A \cos \frac{1}{2}B - \cos \frac{1}{2}A \sin \frac{1}{2}B$ by $\cos \frac{1}{2}(A - B) = \cos \frac{1}{2}A \cos \frac{1}{2}B + \sin \frac{1}{2}A \sin \frac{1}{2}B$, member by member, then proceed in a manner similar to that employed in this article in deriving (41) and thus deduce formula (47).

(*b*) Derive formula (48) by dividing $\sin \frac{1}{2}(A + B)$ by $\cos \frac{1}{2}(A + B)$.

5. (*a*) Divide $\sin \frac{1}{2}(A - B)$ by $\cos \frac{1}{2}(A + B)$ and proceed in a manner similar to that outlined in Exercise 4(*a*) and derive the formula

$$\frac{\sin \tfrac{1}{2}(A - B)}{\cos \tfrac{1}{2}(A + B)} = \frac{\sin \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cos \tfrac{1}{2}c \cot \tfrac{1}{2}C.$$

11-8. Use of Napier's analogies in solving triangles. Napier's analogies are used when the given parts of the triangle are two sides and the included angle or two angles and the side common to them. If the law of sines is used to find the last unknown after two unknowns have been found, often the ambiguity arising may be removed by using the theorem that states that the order of magnitude of the sides of a spherical triangle is the same as that of their respective opposite angles. Other sets of formulas may be obtained from those in Art. 11-7 by the interchange of letters. For example, another set would result from replacing a by c, c by a, A by C, and C by A.

Example. Find A, B, and c for a spherical triangle in which $a = 57°57'$, $b = 137°21'$, $C = 94°48'$.

Solution. In this example $\frac{1}{2}(b - a) = 39°42'$,

$$\tfrac{1}{2}(b + a) = 97°39',$$

$\frac{1}{2}C = 47°24'$. The formulas to be used are

$$\tan \tfrac{1}{2}(B - A) = \frac{\sin \frac{1}{2}(b - a)}{\sin \frac{1}{2}(b + a)} \cot \tfrac{1}{2}C, \tag{a}$$

$$\tan \tfrac{1}{2}(B + A) = \frac{\cos \frac{1}{2}(b - a)}{\cos \frac{1}{2}(b + a)} \cot \tfrac{1}{2}C, \tag{b}$$

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a), \tag{c}$$

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(B + A)}{\cos \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b + a). \tag{d}$$

The following form indicates the computation. The letter in parentheses above each column refers to the formula associated with the column.

	(a)	(b)	(c)	(d) (check)
$\frac{1}{2}(b - a) = 39°42'$	l sin 9.8053	l cos 9.8862	l tan 9.9192	
$\frac{1}{2}(b + a) = 97°39'$	col sin 0.0039	col cos (−)0.8758		l tan (−)0.8719
$\frac{1}{2}C = 47°24'$	l cot 9.9636	l cot 9.9636		
$\frac{1}{2}(B - A) = 30°39.7'$	l tan 9.7728		col sin 0.2925	col cos 0.0654
$\frac{1}{2}(B + A) = 100°39'$		l tan (−)0.7256	l sin 9.9924	l cos (−)9.2667
$\frac{1}{2}c = 57°59.7'$			l tan 0.2041	l tan 0.2040

$A = 69°59'$ $B = 131°19'$ $C = 115°59'$

These results might have been checked by the law of sines.

EXERCISES 11-6

1. Using Napier's analogies, solve the following spherical triangles:

(a) $c = 116°0'$, $A = 70°0'$, $B = 131°18'$.	(b) $a = 88°38'$, $c = 125°18'$, $B = 102°17'$.	(c) $a = 76°24'$, $b = 58°19'$, $C = 116°30'$.
(d) $a = 86°19'$, $b = 45°36'$, $C = 120°46'$.	(e) $a = 41°6'$, $b = 119°24'$, $C = 162°23'$.	(f) $c = 120°19'$, $A = 27°22'$, $B = 91°26'$.

2. In the following, find the angles by means of Napier's analogies and the required side by the law of sines:

(a) $a = 42°45'$, $b = 47°15'$, $C = 11°12'$.

(b) $a = 131°15'$, $b = 129°20'$, $C = 103°37'$.

11-9. Solution of a spherical triangle in which two of the given parts are opposites. Double solutions. For convenience of reference, a theorem from solid geometry is repeated here:

The order of magnitude of the sides of a spherical triangle is the same as that of their respective opposite angles. Or if a and b are a pair of sides of a spherical triangle and A and B the respective opposite angles, we know that if

$$a < b, \qquad \text{then} \qquad A < B. \tag{49}$$

When the given parts of a spherical triangle are two sides and an angle opposite one of them, say, a, b, and A, the angle B may be found by using the law of sines,

$$\sin B = \frac{\sin b}{\sin a} \sin A. \tag{50}$$

Since $\sin B$ does not exceed 1 in magnitude, $\log \sin B$ does not exceed zero. Hence no solution will exist when $\log \sin B > 0$.

When $\log \sin B < 0$, a positive acute angle and its supplement must be considered for B. Each value of B must be consistent with (49). Hence, there will be no solution, one solution, or two solutions according as (49) is satisfied by neither, by one and only one, or by both of the values of B obtained from (50). If $b = a$, then $B = A$, and there is one solution.

Accordingly, begin the solution of a spherical triangle in which a, b, and A are the given parts by using (50) to find $\log \sin B$. If $\log \sin B > 0$, there is no solution. If $\log \sin B < 0$, find two values of B, one a positive acute angle and the other its supplement. Then, to find c and C, use the given parts with each value of B that satisfies (49) in

$$\left.\begin{aligned} \tan \tfrac{1}{2}c &= \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b), \\ \cot \tfrac{1}{2}C &= \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B). \end{aligned}\right\} \tag{51}$$

These formulas were obtained by solving Napier's analogies (41) and (47) for $\tan \frac{1}{2}c$ and $\cot \frac{1}{2}C$, respectively.

A similar discussion, with the roles of sides and angles interchanged, applies when the given parts are two angles and a side opposite one of them; (50) solved for sin b would first be used and then (51).

Example. Given $a = 52°45'$, $b = 71°12'$, $A = 46°22'$. Find c, B, C.

Solution. Two solutions are to be expected. First use the law of sines to find B, and then Napier's analogies to find c_1, C_1, c_2, and C_2. The solution follows.

$a = 52°45'$	col sin 0.0991
$b = 71°12'$	l sin 9.9762
$A = 46°22'$	l sin 9.8596
$B_1 = \mathbf{59°24'}$	l sin 9.9349
$B_2 = \mathbf{120°36'}$	

$\frac{1}{2}(B_1 - A) = 6°31'$	col sin 0.9450	
$\frac{1}{2}(B_1 + A) = 52°53'$	l sin 9.9017	l tan 0.1211
$\frac{1}{2}(b - a) = 9°13'$	l tan 9.2102	col cos 0.0056
$\frac{1}{2}(b + a) = 61°59'$		l cos 9.6719
$\frac{1}{2}c_1 = 48°45'$	l tan 0.0569	
$c_1 = \mathbf{97°30'}$		
$\frac{1}{2}C_1 = 57°50'$		l cot 9.7986
$C_1 = \mathbf{115°40'}$		

$\frac{1}{2}(b - a) = 9°13'$	l tan 9.2102	col cos 0.0056
$\frac{1}{2}(b + a) = 61°59'$		l cos 9.6719
$\frac{1}{2}(B_2 - A) = 37°7'$	col sin 0.2194	
$\frac{1}{2}(B_2 + A) = 83°29'$	l sin 9.9972	l tan 0.9422
$\frac{1}{2}c_2 = 14°56'$	l tan 9.4268	
$c_2 = \mathbf{29°52'}$		
$\frac{1}{2}C_2 = 13°30'$		l cot 0.6197
$C_2 = \mathbf{27°0'}$		

This solution may be checked by the law of sines.

EXERCISES 11-7

Solve the following spherical triangles:

1. $a = 68°53'$, $b = 56°50'$, $B = 45°15'$.

2. $a = 34°1'$, $A = 61°30'$, $B = 24°31'$.

3. $a = 42°15'$, $A = 36°20'$, $B = 46°31'$.

4. $b = 80°$,	**5.** $a = 59°29'$,	**6.** $a = 63°30'$,
$A = 70°$,	$A = 52°51'$,	$b = 132°15'$,
$B = 120°$.	$B = 66°7'$.	$C = 61°18'$.

11-10. Course and distance. In general, the procedure of applying spherical trigonometry to solve problems relating to the

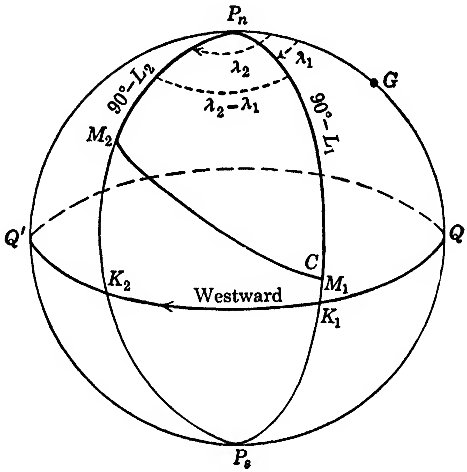


FIG. 11-5.

earth consists in finding three parts of the terrestrial triangle, solving for one or more of the other three parts, and interpreting the results. Consider, for example, the problem of finding the great-circle distance between two points M_1 and M_2 when the latitude and the longitude of each point are known. In Fig. 11-5, P_n represents the north pole, QK_1K_2Q' the equator, P_nGQP_s the meridian of Greenwich, and M_1 and M_2 two places on the earth. The longitudes λ_1 of M_1 and λ_2 of M_2 are known; hence angle

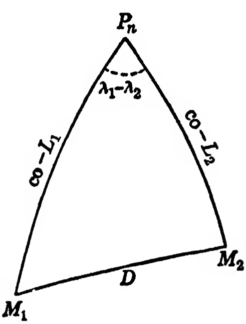


FIG. 11-6.

$$M_1P_nM_2 = \lambda_2 - \lambda_1$$

is known. Also, the latitudes $L_1 = K_1M_1$ of M_1 and $L_2 = K_2M_2$ of M_2 are known; hence the arcs $M_1P_n = 90° - L_1 = co\text{-}L_1$ and $M_2P_n = 90° - L_2 = co\text{-}L_2$ are known. Thus, in triangle

$M_1P_nM_2$, two sides $M_1P_n = co\text{-}L_1$ and $M_2P_n = co\text{-}L_2$ and the included angle $M_1P_nM_2 = \lambda_2 - \lambda_1$ are known. We can solve this triangle by the law of cosines or by Napier's analogies.

EXERCISES 11-8

1. Find the initial course and distance in nautical miles for a great-circle voyage from San Diego (32°43′ N., 117°10′ W.) to Hong Kong (22°9′ N., 114°10′ E.).

2. A ship sails from San Francisco (37°48′ N., 123°23′ W.) to Manila (14°35′ N., 120°58′ E.), following a great-circle track. Find the course angle at departure, the course angle at arrival, and the distance traveled.

3. Find the initial course and the distance for a voyage along a great-circle from Los Angeles (34° 3′ N., 118° 15′ W.) to Wellington (41° 18′ S., 174° 51′ E.).

4. The great-circle distance from Cape Flattery (48° 24′ N., 124° 44′ W.) to Tutuila (14°18′ S., 170°42′ W.) is 4633.7 miles. Find the course of the ship on arrival at Tutuila if it follows a great-circle track from Cape Flattery to Tutuila.

5. Find the distance by great circle from New York (40°40′ N., 73°58′ W.) to Cape of Good Hope (34°22′ S., 18°30′ E.).

6. Find the distance in nautical miles between

		Latitude	Longitude
(*a*)	San Francisco and Honolulu	37°48′ N. 21°18′ N.	122°26′ W. 157°52′ W.
(*b*)	Seattle and Manila	47°36′ N. 14°35′ N.	122°20′ W. 120°58′ E.
(*c*)	Halifax and Cape Town	44°40′ N. 33°56′ S.	63°35′ W. 18°26′ E.
(*d*)	New York and Paris	40°43′ N. 48°50′ N.	73°58′ W. 2°20′ E.
(*e*)	Seattle and Tokyo	47°36′ N. 35°39′ N.	122°20′ W. 139°45′ E.

CHAPTER 12

THE CELESTIAL SPHERE

12-1. Foreword. This chapter deals mainly with apparent motions of the celestial bodies consisting of the sun, moon, planets, and stars. The astronomical triangle with points on the celestial sphere as vertices will play the main role. The formulas developed in other chapters will be applied in the solution of the astronomical triangle.

12-2. The celestial sphere. Consider a fixed star so far away from our solar system that the light rays coming to us from this star appear to follow parallel lines independent of our position; for example, light rays coming from this star to us at one position of the earth's orbit appear to have the same direction as light rays coming from the star to us 6 months later when we are on the other side of the orbit of the earth or approximately 186 million miles from the first position. Since, to us, light rays from this star seem to travel in parallel lines, we naturally associate a fixed direction with it.

We shall speak of the **celestial sphere** as a sphere concentric with the earth and having a radius of unlimited length; by this we shall understand that any two parallel lines cut this sphere in the same point, and any two parallel planes cut it in the same great circle. With any point on this sphere is associated a fixed direction; the angular distance between two points on it may be considered, but not an actual distance in miles.

Figure 12-1 represents the celestial sphere with the earth at its center.

The point P_N on the celestial sphere where a line connecting the center of the earth to its north pole cuts the celestial sphere is called the **north celestial pole;** the point P_S diametrically opposite is called the **south celestial pole.**

The plane of the equator of the earth cuts the celestial sphere in the **equinoctial** or **celestial equator.** The **celestial poles** are the poles of the celestial equator.

The point Z (see Fig. 12-2) directly above an observer, that is, the point where a line connecting the center of the earth to an

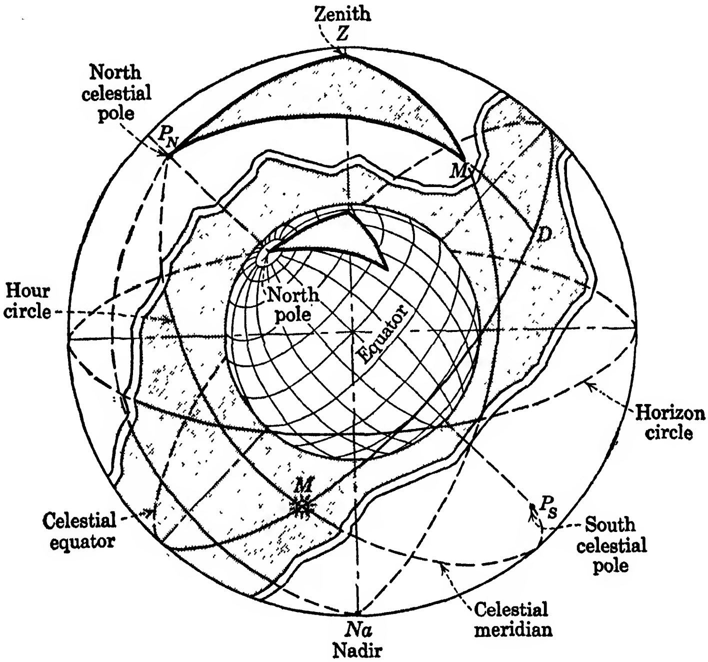


FIG. 12-1.

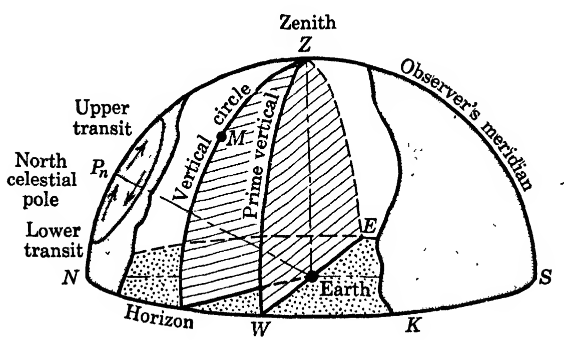


FIG. 12-2.

observer on it would intersect the celestial sphere, is called the **zenith.** The point on the celestial sphere diametrically opposite the zenith is called the **nadir** Na (see Fig. 12-3).

The great circles such as P_NMP_S in Fig. 12-1, passing through the celestial poles, are called **hour circles** or **celestial meridians.** If the meridian in question is that of the observer, the half that contains his zenith and is terminated by the poles is called the upper branch of the meridian.

The **horizon** $NWSE$ of an observer is the great circle on the celestial sphere having the zenith and nadir as poles. A plane tangent to the earth at a point on it intersects the celestial sphere in the celestial horizon associated with the point.

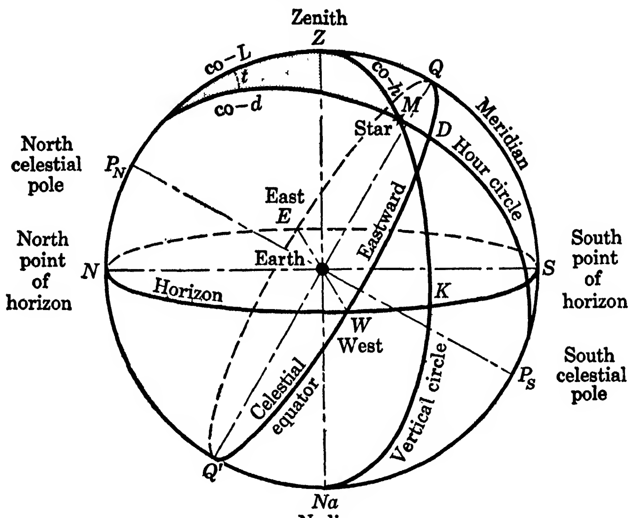


FIG. 12-3.

The point on the horizon directly below the north celestial pole is called the **north point** of the horizon. The **south point,** the **east point,** and the **west point** of the horizon are then determined in the usual way.

The great circles, such as ZMK of the celestial sphere, which pass through the zenith, are called **vertical circles.** Evidently they are all perpendicular to the horizon. The **prime vertical** is the vertical circle EZW (see Fig. 12-2) passing through the zenith and the east and west points of the horizon.

12-3. The astronomical triangle. *The spherical triangle* (see Fig. 12-4) *whose vertices are the north celestial pole, the zenith, and the projection of a heavenly body on the celestial sphere is called*

the astronomical triangle. The solution of many of the problems of astronomy and of navigation requires the solution of this triangle.

The great-circle distance of a point on the celestial sphere from the celestial equator is called the **declination** d of the point. This corresponds to the latitude of a point on the earth. Inspection of Fig. 12-3 shows that the arc P_NM of the astronomical triangle is 90° minus declination, or *co-d.*

The **altitude** h of a point on the celestial sphere is its great-circle distance from the horizon. Inspection of Fig. 12-3 shows that the altitude KM of M is h, and the side MZ of the astronomical triangle is equal to $90° - h$ or *co-h.*

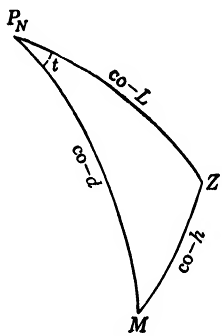


FIG. 12-4.

The azimuth Z_N of a point on the celestial sphere is the angle at the zenith measured eastward from the upper branch of the celestial meridian of the observer to the vertical circle of the point. Thus a body in the eastern sky has an azimuth less than 180°; one in the western sky has an azimuth greater than 180°.

The azimuth angle Z is the angle P_NZM in Fig. 12-4. It is always less than 180°. The arc $NESK$ in Fig. 12-3 measures the azimuth of point M and the arc NWK measures the azimuth angle of point M. Observe that when a body is in the eastern sky, its azimuth is the same as its azimuth angle; when it is in the western sky, its azimuth is 360° minus its azimuth angle. An observer whose latitude is south generally measures his azimuth angle from the southern branch of his meridian. A study of Fig. 12-4 indicates how the azimuth is to be obtained in this case. Evidently the length of P_NZ of the astronomical triangle is 90° minus the observer's latitude or $90° - L$ or *co-L.*

The hour angle at a place of a celestial body is the angle at the elevated pole generated westward from the upper branch of the meridian of the place to the meridian of the body and is measured from 0° to 360°, or from 0 to 24^h.

The hour angle at the position of the observer is referred to as the local hour angle and is designated L.H.A. The hour angle at Greenwich is called the Greenwich hour angle and is designated G.H.A.

As the earth turns on its axis making a complete revolution each day, the heavenly bodies appear to move on the celestial sphere. Thus, the angle through which the earth must turn to bring the celestial meridian of an observer into coincidence with the hour circle of a point on the celestial sphere appears as the hour angle of the point relative to the observer. The angle thus associated with the time of earth revolution is called the meridian angle, t.

When the L.H.A. of a body is less than 180°, the body is in the western sky and t = (L.H.A.) W. Thus when L.H.A. = 100°, t = 100° W. When the L.H.A. of a body is greater than 180°, the body is in the eastern sky and t = (360° − L.H.A.) E. Thus, when L.H.A. = 230°, t = (360° − 230°) E. = 130° E.

EXERCISES 12-1

1. In Fig. 12-5, M represents the position of a star on the celestial sphere, P_n the north celestial pole, Z the observer's zenith, and G the zenith of Greenwich. On this sphere, draw and label a line representing

(a) The celestial meridian of M.
(b) The celestial meridian of G.
(c) The equinoctial.
(d) The horizon circle.

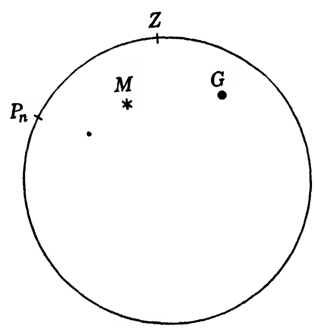


FIG. 12-5.

2. Place the letters N, A, and E on the sphere in Fig. 12-5 such that PN represents the latitude of the observer, AM the altitude of M, and EM the declination of M.

3. What angle in Fig. 12-5 represents

(a) The longitude of the observer?
(b) The G.H.A. of M?
(c) The L.H.A. of M?
(d) The meridian angle t?
(e) The azimuth angle of M?

4. Identify the astronomical triangle for the star M in Fig. 12-5. Label its sides and two useful angles.

5. When the sun is on the horizon, what is its zenith distance? When it is on the equinoctial, what is its declination?

6. For each of the following local hour angles, find the meridian angle t. Draw a figure for each case.

(a) 48°.	(b) 142°.	(c) 217°.
(d) 332°.	(e) 467°.	(f) 594°.

7. For each of the following azimuth angles (Z) find the azimuth (Z_n):

(a) N. 34° E.	(b) S. 54° W.	(c) N. 127° W.
(d) S. 97° E.	(e) N. 127° E.	(f) N. 64° W.

8. Convert the following true azimuths (Z_n) to azimuth angles (Z):

(a) North latitude, $Z_n = 123°$.
(b) South latitude, $Z_n = 264°$.
(c) South latitude, $Z_n = 145°$.
(d) North latitude, $Z_n = 359°$.

9. The L.H.A. and declination of a heavenly body are 342° and 27° S., respectively. If the latitude of the observer is 50° N., find the two sides and the included angle t of the astronomical triangle associated with these data.

10. A navigator in latitude 42° N. observes a star and obtains the following data: true altitude 40°, declination 27° N. Find the sides of the astronomical triangle associated with the observation.

11. An observer is in longitude 40° W. What is the G.H.A. of his zenith?

12-4. Given *t*, *d*, *L*. Find *h* and *Z*. Figure 12-6 represents the astronomical triangle with the given parts encircled. Since

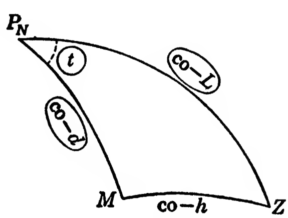


FIG. 12-6.

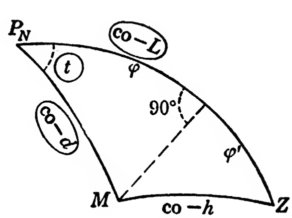


FIG. 12-7.

two sides and the included angle are given, we may solve for the unknown parts by the methods developed in Arts. 11-3 and 11-8, the law of cosines and Napier's analogies. Or we may construct

an arc of a great circle through M perpendicular to $P_N Z$, letter the triangle as shown in Fig. 12-7, and then apply Napier's rules to obtain

$$\begin{aligned}
\tan &= \cos t \cot d, \\
\varphi' &= 90^\circ - L - \varphi = 90^\circ - (L + \varphi), \\
\cot Z &= \cot t \sin \varphi' \csc \varphi = \cot t \cos (L + \varphi) \csc \varphi, \\
\sin h &= \cos \varphi' \sec \varphi \sin d = \sin (L + \varphi) \sec \varphi \sin d, \\
\sin t \cos d &\csc Z \sec h = 1. \quad (\textit{Check})
\end{aligned}$$

If L represents the latitude of a place north of the equator, d should be taken positive for a body having north declination and negative for one having south declination, or vice versa.

EXERCISES 12-2

In the following exercises, compute h and Z_n:

1. $d = 6^\circ 15'$ S., $t = 14^\circ 6'$ W., $L = 21^\circ 18'$ N.

2. $d = 10^\circ$ N., $t = 40^\circ$ W., $L = 35^\circ$ S.

3. $d = 38^\circ 17'$ S., $t = 28^\circ 31'$ W., $L = 24^\circ 33'$ N.

4. $d = 7^\circ$ S., $t = 28^\circ$ E., $L = 41^\circ$ N.

5. $d = 59^\circ 56'$ N., $t = 60^\circ 32'$ E., $L = 44^\circ 45'$ N.

6. $d = 8^\circ$ N., $t = 35^\circ$ E., $L = 39^\circ$ N.

7. $d = 10^\circ$ S., $t = 25^\circ$ E., $L = 18^\circ 58'$ S.

8. $d = 22^\circ 30'$ S., $t = 60^\circ$ E., $L = 45^\circ$ S.

12-5. The time of day. Owing to the rotation of the earth, the sun appears to move across the sky from east to west. This rate of rotation is almost constant and furnishes a basis for the measurement of time.

Local apparent noon for an observer is the time of day when the sun is on his meridian. During the forenoon the sun appears to move in the eastern sky, and during the afternoon it appears to move in the western sky. At noon it is on the observer's meridian. Thus, in Fig. 12-8, M represents the position of the sun at noon, R represents its position on the horizon NRS at sunset, and S' represents its position at any time in the after-

noon.* Since the earth turns about its axis P_NP_S, angle $MP_NS' = t$, called the meridian angle, measures the time since noon, that is, the time of day. As the sun appears to make a complete circuit of the earth approximately once every 24 hr., we associate

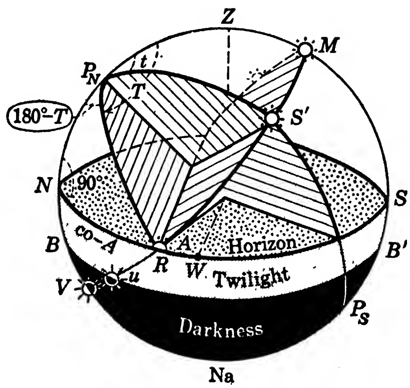


FIG. 12-8.

360° of angle with 24 hours of time, and express the relation between angle and time by writing

24 hours correspond to 360°.	
1 hour corresponds to 15°.	$\therefore\ 1^\circ = \frac{1}{15}$ hr. $= 4$ min.
1 minute corresponds to 15′.	$\therefore\ 1' = \frac{1}{15}$ min. $= 4$ sec.
1 second corresponds to 15″.	$\therefore\ 1'' = \frac{1}{15}$ sec.

For example, $1^{\text{h}}10^{\text{m}}20^{\text{s}} = 1(15^\circ) + 10(15') + 20(15'') = 17^\circ 35'$.

$$\text{Also, } 35^\circ 38' 40'' = \left(\frac{30}{15}\right)^{\text{h}}\left(\frac{5 \times 60}{15}\right)^{\text{m}}\left(\frac{30}{15}\right)^{\text{m}}\left(\frac{8 \times 60}{15}\right)^{\text{s}}\left(\frac{40}{15}\right)^{\text{s}}$$
$$= 2^{\text{h}}22^{\text{m}}34.7^{\text{s}}.$$

The astronomical triangle P_NZS may be solved to find t. The sides of the triangle may be considered as known. The observer knows his latitude L. He measures the altitude h with a sextant and finds the sun's declination d in the Nautical Almanac. He, therefore, knows the sides *co-L*, *co-h*, and *co-d*. The triangle is solved by means of the half-angle formulas developed in Art. 11-5.

* During the forenoon the sun is in the eastern sky and the angle t between the celestial meridian of the sun and that of the observer measures the number of hours before noon.

Example. Find the azimuth Z_N of the sun and the local apparent time in New York (40°43′ N.) at the instant when the altitude of the sun is 30°10′ bearing west and its declination is 10° N.

Solution. Let $A = t$, let $a = co\text{-}h = 90° - 30°10' = 59°50'$, let $b = co\text{-}d = 90° - 10° = 80°$, let

$$c = co\text{-}L = 90° - 40°43' = 49°17',$$

and let $B = Z$. Find A and B as indicated in the following form:

$a = 59°50'$			
$b = 80°$			
$c = 49°17'$			
$s = 94°33.5'$	col sin 0.0013		
$s - a = 34°43.5'$	l sin 9.7556	col sin 0.2444	
$s - b = 14°33.5'$	l sin 9.4003		col sin 0.5997
$s - c = 45°16.5'$	l sin 9.8515		
	2)9.0087		
r	9.5044	log 9.5044	log 9.5044
$\frac{1}{2}A = 29°17'$		l tan 9.7488	
$\frac{1}{2}B = 51°48'$			l tan 0.1041
$A = t = 58°34'$			
$B = Z =$ N. 103°36′ W.			

Hence, $t = \mathbf{3^h54^m}$ and $Z_n = \mathbf{256°24'}$.

EXERCISES 12-3

1. Express in degrees, minutes, and seconds the angle corresponding to each of the following:

(*a*) $3^h15^m18^s$. (*b*) $0^h27^m19^s$. (*c*) $7^h5^m12^s$.
(*d*) $15^h21^m9^s$. (*e*) $23^h56^m34^s$. (*f*) $12^h32^m16^s$.

2. Express in hours, minutes, and seconds the time corresponding to each of the following angles:

(*a*) 120°15′30″. (*b*) 40°27′19″. (*c*) 79°17′16″.
(*d*) 260°34′28″. (*e*) 90°15′35″. (*f*) 332°12′56″.

3. An observation of the altitude of the sun was made in each of the following cities. Find the azimuth of the sun and the local apparent time of observation in each case.

(*a*) Pensacola, Fla., $L = 30°21'$ N., sun's altitude $h = 24°30'$ bearing east, declination $20°42'$ N.

(*b*) Philadelphia, Pa., $L = 40°0'$ N., $h = 26°0'$ E., $d = 20°0'$ N.

(*c*) Annapolis, Md., $L = 39°0'$ N., $h = 22°0'$ E., $d = 20°0'$ N.

Given the following data, find t and Z:

4. $L = 42°45'$ N., $d = 18°27'$ N., $h = 38°36'$ E.

5. $L = 45°0'$ N., $d = 22°30'$ N., $h = 30°0'$ W.

6. $L = 25°35'$ N., $d = 10°24'$ S., $h = 35°19'$ E.

7. $L = 30°0'$ N., $d = 15°0'$ N., $h = 45°0'$ W.

12-6. To find the time and amplitude of sunrise. Figure 12-9 represents a stereographic projection of the astronomical triangle P_NZM when the body M is the sun on the horizon. The dotted line indicates the path of the sun across the sky as a small circle each of whose points is distant *co-d* from the pole. When the sun crosses the meridian at K, it is noon. Hence t represents the angle through which the earth must turn during the time interval from sunrise to noon. Since the earth turns through 15° per

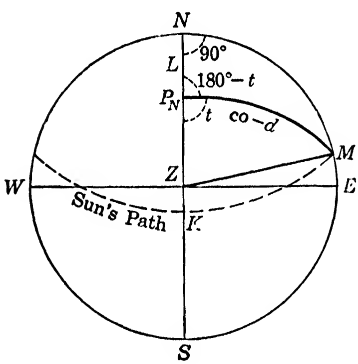


FIG. 12-9.

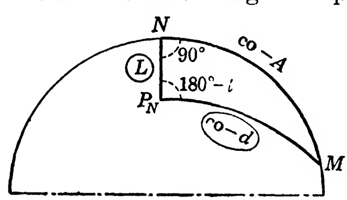


FIG. 12-10.

hour, $t/15$ will be the number of hours from sunrise to noon if t is expressed in degrees. The declination of the sun can be found from the Nautical Almanac, and the latitude of the observer is supposed to be known. Therefore, to find a formula for t, apply Napier's rules to right spherical triangle NMP_N (Fig. 12-10), and write $\cos(180° - t) = \tan d \tan L$, or

$$\cos t = -\tan d \tan L. \qquad (a)$$

The angular distance from the east point of the horizon to the sun at sunrise is called the *amplitude of sunrise.* From

right spherical triangle NP_NM of Fig. 12-10 we find, by using Napier's rules, $\sin d = \cos L \sin A$, or

$$\sin A = \sin d \sec L. \tag{b}$$

From Fig. 12-10 we obtain the check formula

$$-\cot A \cot t \csc L = 1. \tag{c}$$

Example. Find the amplitude and the time of sunrise at Annapolis, $L = 38°59'$ N., at a time when the declination of the sun is 20° S.

Solution. The solution found from formulas (*a*), (*b*), and (*c*) appears below.

	(*a*)		(*b*)		(*c*)	
$L = 38°59'$	l tan	9.9081	col cos	0.1094	col sin	0.2014
$d = -20°0'$	l tan (−1)	9.5611	l sin (−1)	9.5341		
$t = 72°52'$	l cos	9.4692			l cot	9.4890
$A = -26°7'$			l sin (−1)	9.6435	l cot	0.3096
					log	0.0000

Since 15° indicates a time of 1^h, 72°52 will indicate 4^h51^m. As t is the time from sunrise till noon, we obtain

$$12^h - (4^h51^m) = \mathbf{7^h9^m}$$

as the local apparent time* of sunrise. The negative sign before the amplitude indicates that the sun appeared on the horizon *south* of the east point.

EXERCISES 12-4

In Exercises 1-4 assume that sunrise or sunset occurs when the center of the sun is on the horizon.

1. Find the amplitude of sunrise in Lat. 38°59′ N. when the declination of the sun is 22°29′ S.

2. At Annapolis, Lat. 38°59′ N., the sun in declination 23°27′ N. has the altitude 0°, bearing easterly. Find the local apparent time.

* The noon of local apparent time occurs when the sun is on the meridian of the observer, and the time of the day is expressed in terms of the hour angle of the sun. Owing to refraction of the sunbeams by the earth's atmosphere, the sun will appear to be on the horizon considerably earlier than the results of this computation would indicate. In practice corrections must be made on this account.

3. Find the amplitude and the local apparent time of sunrise and sunset for Annapolis, Md., $L = 38°59'$ N., at summer and winter solstice ($d = \pm 23°28'$).

4. (*a*) Find the local apparent time of sunrise and sunset at Cape Nome, $L = 64°23'$ N., on Mar. 21, $d = 0°0'$, Dec. 21, $d = 23°27'$ S., and June 21, $d = 23°27'$ N. (*b*) Find the amplitude of the sun at each occurrence. (*c*) Find the length of the longest day and of the shortest day at Cape Nome.

5. Assuming that the declination of the sun ranges between 23°27′ S. to 23°27′ N., show that a place where the sun rises at midnight must lie within 23°27′ of a pole of the earth.

Hint. In the formula $\cos t = -\tan L \tan d$, let $t = 180° \; (= 12^{\text{h}})$.

6. For a point on the earth having Lat. 80° N. find (*a*) the declination of the sun when the time of daylight is just 24 hr.; (*b*) the declination of the sun when the night lasts just 24 hr.; (*c*) the least altitude and the greatest altitude of the sun during the day when the declination of the sun is 23°27′ N.; (*d*) the declination of the sun when continuous night begins; (*e*) the length of the shortest possible shadow cast by a vertical pole 20 ft. long.

12-7. Meridian altitude. To find the latitude of a place on the earth. Figure 12-11 represents the cross section of the earth

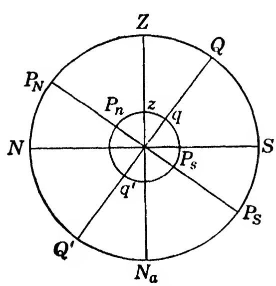


FIG. 12-11.

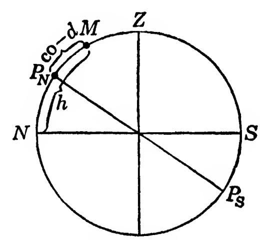


FIG. 12-12.

and of the surrounding celestial sphere by the plane of the meridian of an observer. qq' represents the equator of the earth; z, the position of the observer; and P_nP_s, the axis of the earth. QQ', Z, P_NP_S, N, and S represent, respectively, the celestial equator, the zenith, axis of celestial sphere, north point of the horizon, and south point of the horizon. Since qz represents the latitude of the observer and since arc qz = arc QZ = arc NP_N, it appears that the **latitude of an observer on the earth is**

equal to the declination of his zenith and to the altitude of the pole elevated above his horizon.

If, then, an observer knows the declination d* of a star M (see Fig. 12-12) and observes its altitude h† just as it crosses his meridian above the pole, he can find his latitude by writing

$$L = NP_N = h - (90° - d).$$

The student should draw a figure for each case. First, a figure like Fig. 12-12 should be drawn showing the circle, Z, N, and S. Then the star M should be located on the figure so that arc $NM = h$ if the star bears north or so that $SM = h$ if it bears south.

Next, the pole should be located so that arc

$$MP_N(\text{or } MP_S) = 90° - d.$$

Finally, the altitude of the pole elevated above the horizon should be computed from the figure.

Example. Find L if the declination of a star is 62° S. and if its altitude as it crosses the meridian at upper culmination‡ is 50° bearing south.

Solution. Since the star bears south and since it appears in the sky 50° above the horizon, it is represented in Fig. 12-13 on the right side of the circle so that arc $SM = 50°$. Next

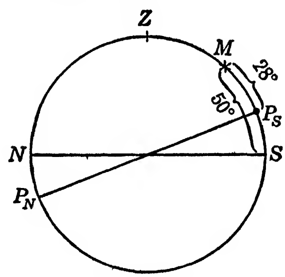


FIG. 12-13.

$$MP_S = 90° - d = 90° - 62° = 28°$$

is laid off to locate P_S. Hence the latitude is

$$L = 50° - 28° = \mathbf{22°\ S.}$$

* The declination of a star can be found from the Nautical Almanac.

† Various corrections to the observed altitude are generally necessary to obtain the true altitude.

‡ The stars appear to move through the sky, each describing a small circle, one of whose poles is the celestial north pole, the other, the celestial south pole. Thus each star crosses the plane of the meridian of a place twice every 24 hr., the first time on one side of the pole and the second time on the opposite side. The greater of the two altitudes of meridian transit is the altitude of upper culmination; the lesser is the altitude of lower culmination.

The observer must have been in south latitude since the south pole was elevated above the horizon.

EXERCISES 12-5

From the meridian altitude h, the declination d, and the bearing of the observed body as indicated, find the latitude of the observer in each of the following cases

Assume in each of the Exercises 1 to 12 that the body is in upper culmination.

	d	h		d	h
1.	50° N.	40° N.	**7.**	41°39′ N.	82°11′ N.
2.	40° S.	20° S.	**8.**	37°15′ N.	40°21′ N.
3.	20° N.	60° S.	**9.**	11°0′ N.	70°19′ N.
4.	50°25′ S.	35°29′ S.	**10.**	17°39′ S.	72°21′ S.
5.	30°15′ S.	47°35′ N.	**11.**	47°23′ S.	35°26′ S.
6.	28°10′ N.	71°12′ S.	**12.**	23°13′ N.	75°40′ S.

Assume in each of the Exercises 13 to 16 that the body is in lower culmination.

13.	59°49′ N.	44°11′ N.	**15.**	73°16′ N.	28°48′ N.
14.	77°54′ S.	25°18′ S.	**16.**	42°29′ N.	25°23′ S.

17. Two observers, A and B, are at different places on the same meridian. At the same instant each observer measured the meridian altitude of a star having declination 26°16′ S. A observed the star bearing south at an altitude 30°17′, B observed the star bearing north at an altitude 60°17′. Find the great-circle distance between A and B.

MISCELLANEOUS EXERCISES 12-6

1. Given $t = 45°10'30''$ W., $d = 1°9'15''$ S., $L = 37°30'$ N., find the azimuth Z_n.

2. Given $t = 55°$ E., $d = 15°$ S., and $L = 42°$ N., find h and Z.

3. Given $t = 30°$ W., $d = 45°$ N., $h = 60°$, find L and Z.

4. Given $t = 30°$ E., $d = 15°$ S., $h = 60°$, find L and Z.

5. From the following data, compute in each case the latitude and azimuth:

(*a*) $h = 68°$,
$t = 10°$ E.,
$d = 23°$ S.

(*b*) $t = 30°11'$ E.,
$d = 22°29'$ N.,
$h = 44°57'$.

6. In each of the following exercises, L represents the latitude of the observer, d the declination of a star, and h its altitude. Find in each case the hour angle t and the azimuth Z_n of the star.

(*a*) $L = 45°$ N., $d = 22°30'$ N., $h = 30°$ W.
(*b*) $L = 30°$ S., $d = 15°$ N., $h = 37°30'$ E.

7. An airplane following a great-circle track travels from a place having $L = 37°50'$ N., $\lambda = 122°20'$ W. (near Oakland, Calif.) to a place having $L = 40°40'$ N., $\lambda = 74°10'$ W. (near Newark, N. J.). How close does it pass to a point for which $L = 41°50'$ N., $\lambda = 87°40'$ W. (near Chicago, Ill.)?

8. Compute the distance and the initial course for a voyage along a great circle from Yokohoma, $L = 35°27'$ N., $\lambda = 139°39'$ E., to Diamond Head, Hawaii, $L = 21°51'$ N., $\lambda = 157°\ 49'$ W.

9. Compute the distance and the initial course for a voyage along a great circle from Brisbane, Australia, $L = 27°28'$ S., $\lambda = 153°2'$ E., to Acapulco, $L = 16°49'$ N., $\lambda = 99°56'$ W. Also find the latitude and longitude of the southern vertex of the track.

10. Compute the distance and the initial course for a great-circle voyage from a point having $L = 37°42'$ N., $\lambda = 123°4'$ W., near Farallon Island Lighthouse, to a point having $L = 34°50'$ N., $\lambda = 139°53'$ E., near the entrance to the Bay of Tokyo.

11. Find the distance and the initial course of a great-circle voyage from San Diego, $L = 32°43'$ N., $\lambda = 117°10'$ W., to Cavite, $L = 14°30'$ N., $\lambda = 120°55'$ E.

12. Find where the track of the preceding exercise crosses the meridian of $157°49'$ W. and at what distance from the harbor of Honolulu, $L = 21°16'$ N., $\lambda = 157°49'$ W., then due south.

13. The initial course by great-circle track from San Francisco, $L = 37°50'$ N., $\lambda = 122°30'$ W., to Yokohama, $L = 35°30'$ N., $\lambda = 140°$ E., is $302°59'$. Find the longitude of the most northerly point of this path.

14. Find the latitude and the longitude of the most northerly point reached by a ship sailing from San Francisco, Lat. 37°48′ N., Long. 122°28′ W., to Calcutta, Lat. 22°53′ N., Long. 88°19′ E.

15. An airplane follows a great-circle track from near New York, $L = 40°40'$ N., $\lambda = 74°10'$ W., to $L = 56°30'$ N., $\lambda = 3°0'$ W. (near Edinburgh, Scotland). Where will it make its nearest approach (*a*) to the north pole? (*b*) To $L = 46°40'$ N., $\lambda = 71°10'$ W. (near Quebec, Canada)?

16. Find the local apparent time of sunrise and sunset at

(*a*) London: $L = 51°29'$ N., if d of sun $= 13°17'$ N.
(*b*) Panama: $L = 8°57'$ N., if d of sun $= 18°29'$ N.
(*c*) New Orleans: $L = 29°58'$ N., if d of sun $= 4°30'$ N.
(*d*) Sydney: $L = 33°52'$ S., if d of sun $= 4°30'$ N.

17. Find the length (*a*) of the longest day; (*b*) of the shortest day at Leningrad $L = 59°57'$ N., $\lambda = 30°19'$ E.

18. The following observations have been made of a heavenly body in upper culmination. Find the latitude in each case.

	Declination	Observed altitude	Bearing
(*a*)	28°10′ N.	71°21′	South
(*b*)	73°02′ N.	58°40′	North
(*c*)	44°17′ S.	65°23′	South
(*d*)	30°15′ S.	47°35′	North
(*e*)	50°25′ S.	35°29′	South
(*f*)	40°16′ N.	40°14′	North

19. In each of the following observations of a lower culmination, find the latitude:

	Declination	Observed altitude	Bearing
(*a*)	88°50′ N.	37°20′	North
(*b*)	46°22′ S.	32°15′	South
(*c*)	59°49′ N.	44°11′	North
(*d*)	77°54′ S.	25°18′	South

20. At a place in Lat. 51°32′ N., the altitude of the sun is 35°15′ bearing west and its declination is 21°27′ N. Find the local apparent time.

21. In London, $L = 51°31'$ N., for an afternoon observation the altitude of the sun is 15°40′. If its declination is 12° S., find the local apparent time.

CHAPTER 13

LOGARITHMS

13-1. Introduction. The labor involved in many numerical computations is considerably lessened by the use of logarithms. In the following articles we shall discover that, in a sense, the use of logarithms reduces multiplication to addition, division to subtraction, raising to a power to multiplication, and extracting a root to division. For this reason logarithms constitute a remarkable labor-saving device in computation.

We shall learn presently that logarithms are exponents and that the laws that govern the use of exponents are the ones that govern the use of logarithms. Hence, before discussing logarithms, we shall recall from algebra the laws of exponents.

13-2. Laws of exponents. It is proved in algebra that, when the exponents m and n are any numbers, the following laws hold:

(I) $a^m a^n = a^{m+n}$. (II) $\dfrac{a^m}{a^n} = a^{m-n}$.

(III) $(a^m)^n = a^{mn}$. (IV) $(ab)^m = a^m b^m$.

(V) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

EXERCISES 13-1

1. Evaluate the following:

(*a*) $3^2 3^{-3}$. (*b*) $7^{-\frac{3}{7}} \sqrt[7]{7^{10}}$. (*c*) $3^{-\frac{1}{2}} 3^0$.

(*d*) $3^{-\frac{3}{2}} 3^{\frac{7}{2}}$. (*e*) $\dfrac{5^{-\frac{3}{2}}}{\sqrt{5}}$. (*f*) $(3^{-1})^{\frac{27}{8}}$.

(*g*) $(25 \times 49)^{-\frac{1}{2}}$. (*h*) $(\frac{3}{2})^{-3}$. (*i*) $(\frac{8}{27})^{-\frac{2}{3}}$.

2. Find, in each case, the value of x that satisfies the equation:

(*a*) $10^x = 1000$. (*b*) $3^{-3} = x$. (*c*) $x^4 = 10{,}000$.
(*d*) $x^{-\frac{1}{2}} = 3$. (*e*) $4^x = \frac{1}{2}$. (*f*) $x^{-2} = 100$.
(*g*) $10^0 = x$. (*h*) $x^{-2} = 100$. (*i*) $(36)^x = \frac{1}{6}$.

(*j*) $x^{-\frac{1}{3}} = \sqrt{7}$. (*k*) $7^x = 1$. (*l*) $x^{-1} = 0.01$.
(*m*) $7^x = 343$. (*n*) $\left(\frac{1}{x}\right)^{-2} = 16$. (*o*) $2^{\frac{1}{x}} = 4^3$.

3. Find x if

(*a*) $10^x = \frac{1}{10}$. (*b*) $10^x = 0.001$.
(*c*) $10^x = 0.0001$. (*d*) $10^x = 1000$.
(*e*) $10^x = 1$. (*f*) $10^x = 100{,}000$.

4. Solve each of the following equations for x:

(*a*) $(3)(2)^x + 4 = 100$. (*b*) $5^{x+3} - 5^{2x} = 0$.
(*c*) $(8)(2)^x - 2^{4x} = 0$. (*d*) $(8)(3^x) = (27)(2^x)$.
(*e*) $(x - 2)^0 = x^2 + 1$. (*f*) $27^x = 81$.
(*g*) $(3^{\frac{1}{2}})(9)^{2x} = 3^{-\frac{2}{3}}$. (*h*) $(\frac{16}{25})^{-\frac{1}{2}} = 5\sqrt{x}$.
(*i*) $(\frac{8}{27})^{-\frac{1}{3}} = 2x^{-1}$. (*j*) $(7^{x^2-1})(49^{1-x}) = \sqrt{7}$.
(*k*) $\left(\frac{9x}{4}\right)^{-\frac{1}{2}} - 3^{-2} = 3^{-3}$. (*l*) $\frac{1}{2}\sqrt{x}\sqrt[3]{x} = 64$.

13-3. Definition of a logarithm. If b, L, and N are numbers such that b raised to the power L is equal to N, then L is called the logarithm of N to the base b. In symbols, if

$$b^L = N, \qquad \text{then} \qquad L = \log_b N. \tag{1}$$

Stated differently, **the logarithm of a number to a given base is the power to which the base must be raised to produce the number.**

The two equations in (1) express the same relation between the base b, the number N, and the logarithm L. The second one is read: L is the logarithm of N to the base b. Also N is called the antilogarithm of L (or the number whose logarithm is L) to the base b. Since $5^2 = 25$, 2 is the logarithm of 25 to the base 5, and 25 is the antilogarithm of 2 to the base 5. Similarly, we have

$$\begin{aligned} 10^3 &= 1000, & \therefore\quad 3 &= \log_{10} 1000; \\ 10^{-2} &= 0.01, & \therefore\quad -2 &= \log_{10} 0.01; \\ 3^{\frac{1}{2}} &= \sqrt{3}, & \therefore\quad \tfrac{1}{2} &= \log_3 \sqrt{3}. \end{aligned}$$

Since $1^x = 1$ for all values of x, 1 cannot be used as a base for logarithms. Also a negative number is not used as base; for many real numbers would have imaginary logarithms to a nega-

tive base. For example, if $(-3)^x = 27$, x is imaginary. Although any positive number different from 1 might be used as a base, 10 is often chosen for reasons that will appear as our study continues.

EXERCISES 13-2

Write each of the following exponential equations as a logarithmic equation:

1. $2^4 = 16$. **2.** $10^2 = 100$. **3.** $\sqrt{100} = 10$.
4. $(\frac{1}{2})^{-2} = 4$. **5.** $8^{\frac{2}{3}} = 4$. **6.** $10^{-2} = 0.01$.
7. $25^{-\frac{1}{2}} = \frac{1}{5}$. **8.** $10^0 = 1$. **9.** $10^{-3} = 0.001$.

Write each of the following equations as an exponential equation:

10. $\log_2 8 = 3$. **11.** $\log_5 1 = 0$. **12.** $\log_7 49 = 2$.
13. $\log_{10} 0.1 = -1$. **14.** $\log_9 \frac{1}{3} = -\frac{1}{2}$. **15.** $\log_9 1 = 0$.

In each of the following exercises, find the value of x:

16. $\log_6 x = 2$. **17.** $\log_x \frac{1}{4} = 2$. **18.** $\log_5 25 = x$.
19. $\log_x 15 = 1$. **20.** $\log_2 x = 3$. **21.** $\log_2 x = -2$.
22. $\log_4 x = -\frac{1}{2}$. **23.** $\log_{10} 100 = x$. **24.** $\log_2 32 = x$.
25. $\log_5 (\frac{1}{625}) = x$. **26.** $\log_{10} x = 2$. **27.** $\log_{10} x = -2$.
28. $\log_x 3 = -\frac{1}{2}$. **29.** $\log_x 49 = -2$. **30.** $\log_x 49 = 2$.
31. $\log_{27} 3 = x$. **32.** $\log_2 \left(\frac{1}{\sqrt[3]{16}}\right) = x$. **33.** $\log_5 x = 1$.
34. $\log_b x = 1$. **35.** $\log_x (\frac{1}{9}) = 2$. **36.** $\log_b x = 0$.

Show that

37. $(\log_b a)(\log_a b) = 1$.

38. $(\log_b a)(\log_c b)(\log_a c) = 1$.

39. $\log_b \left(\frac{1}{b}\right) = -1$.

40. Why cannot unity be used as a base for a system of logarithms?

41. Why cannot a negative number be used as a base for a system of logarithms?

13-4. Laws of logarithms. There are three fundamental laws of logarithms with which the student must be thoroughly familiar. These laws are easily derived from the laws of exponents.

I. The logarithm of the product of two numbers is equal to the sum of the logarithms of the factors.

Proof. Let M and N be any two positive numbers, and let

$$x = \log_b N, \qquad \text{and} \qquad y = \log_b M. \tag{2}$$

Then we may write

$$b^x = N, \qquad \text{and} \qquad b^y = M. \tag{3}$$

Multiplying, member by member, the first of equations (3) by the second, we get

$$b^x b^y = b^{x+y} = MN, \qquad \text{or} \qquad \log_b MN = x + y. \tag{4}$$

Substituting the values of x and y from (2) in (4), we get

$$\mathbf{\log_b MN = \log_b M + \log_b N.}$$

By repeated application of the first law it is readily proved that **the logarithm of the product of any finite number of factors is equal to the sum of the logarithms of the factors.**

II. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

Proof. Dividing, member by member, the first of equations (3) by the second, we get

$$\frac{N}{M} = \frac{b^x}{b^y} = b^{x-y}, \qquad \text{or} \qquad \log_b \frac{N}{M} = x - y. \tag{5}$$

Substituting the values of x and y from (2) in (5), we get

$$\mathbf{\log_b \frac{N}{M} = \log_b N - \log_b M.}$$

III. The logarithm of a number affected by an exponent is the product of the exponent and the logarithm of the number.

Proof. Let

$$x = \log_b N, \qquad \text{or} \qquad N = b^x. \tag{6}$$

Raising both members of $N = b^x$ to the pth power, we obtain

$$N^p = b^{px},$$

Therefore, in accordance with (1)

$$\log_b N^p = px. \tag{7}$$

Substitution of the value of x from (6) in (7) gives

$$\mathbf{\log_b N^p = p \log_b N.}$$

Example 1. Find the value of $\log_{10} \sqrt{0.001}$.

Solution. $\log_{10} \sqrt{0.001} = \log_{10} (0.001)^{\frac{1}{2}} = \frac{1}{2} \log_{10} 0.001$
$= \frac{1}{2} \log_{10} \frac{1}{1000} = \frac{1}{2}(-3) = -\frac{3}{2}.$

Example 2. Write $\log_b \sqrt[3]{\dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}}$ in expanded form.

Solution. $\log_b \sqrt[3]{\dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}} = \frac{1}{3} \log_b \dfrac{a^2(c+d)^{\frac{1}{2}}}{c^5}$

$= \frac{1}{3}[\log_b a^2 + \log_b (c+d)^{\frac{1}{2}} - \log_b c^5]$

$= \mathbf{\frac{1}{3}[2 \log_b a + \frac{1}{2} \log_b (c+d) - 5 \log_b c].}$

Example 3. Write $\frac{3}{2} \log_b (x+1) + \frac{1}{3} \log_b x - 2 \log_b (x^2+1)$ in contracted form.

Solution. $\frac{3}{2} \log_b (x+1) + \frac{1}{3} \log_b x - 2 \log_b (x^2+1)$

$= \log_b (x+1)^{\frac{3}{2}} + \log_b x^{\frac{1}{3}} - \log_b (x^2+1)^2$

$= \mathbf{\log_b \dfrac{(x+1)^{\frac{3}{2}} x^{\frac{1}{3}}}{(x^2+1)^2}}.$

Another form of the answer is found as follows:

$$\log_b \frac{(x+1)^{\frac{3}{2}} x^{\frac{1}{3}}}{(x^2+1)^2} = \log_b \left[\frac{(x+1)^9 x^2}{(x^2+1)^{12}}\right]^{\frac{1}{6}} = \mathbf{\frac{1}{6} \log_b \frac{(x+1)^9 x^2}{(x^2+1)^{12}}}.$$

EXERCISES 13-3

1. Verify the following:

(*a*) $\log_{10} \sqrt{1000} + \log_{10} \sqrt{0.1} = 1.$
(*b*) $\log_2 \sqrt{8} + \log_2 \sqrt{2} = 2.$
(*c*) $\log_8 (2)^5 + \log_7 (\frac{1}{49})^{\frac{1}{3}} = 1.$
(*d*) $\log_2 \sqrt{8} + \log_3 (\frac{1}{3})^2 = -\frac{1}{2}.$
(*e*) $\log_5 \sqrt{125} + \log_{13} \sqrt[3]{169} = \frac{13}{6}.$
(*f*) $\log_{11} \frac{1}{11} + 2 \log_{11} \sqrt{11} = 0.$
(*g*) $\log_2 (0.5)^3 - \log_4 \sqrt[6]{64} = -\frac{7}{2}.$
(*h*) $\log_5 1 - \log_7 6^0 = 0.$
(*i*) $\log_{10} 10^5 - \log_{10} 10^2 + \log_{10} 10^{-2} + \log_{10} 1 = 1.$

2. Write the following logarithmic expressions in expanded form:

(a) $\log_b \dfrac{a^2 b^{\frac{1}{2}}}{c^3}$. (b) $\log_b \left(\dfrac{a^3 b^6}{c^2}\right)^{\frac{1}{2}}$. (c) $\log_b \sqrt[5]{\dfrac{a^{\frac{1}{2}} c^{\frac{5}{2}}}{d^7}}$.

(d) $\log_b P(1 + r)^n$. (e) $\log_b \dfrac{a^3 c d^5}{7 \sqrt[4]{e}}$. (f) $\log_b \sqrt[3]{\dfrac{x(x - y)}{z(x + y)}}$.

(g) $\log_b \dfrac{\sqrt[3]{p^2(1 - q)}}{p^{\frac{1}{2}}(1 + q)}$. (h) $\log_b \dfrac{[\sqrt{p - 1}]^3}{q^2}$. (i) $\log_b \left[\dfrac{(p^0 - 5)^{\frac{1}{2}}}{(p - 7)^2}\right]^5$.

(j) $\log_b \dfrac{(x + g)x^2}{\sqrt{x - y}\,(z + y)}$. (k) $\log_b \dfrac{a(c - d)^2}{6(a + f)}$.

(l) $\log_b \sqrt[5]{\left[\dfrac{a^2(c - d)^3}{c\sqrt{a - d}}\right]^2}$.

3. Write the following expressions in contracted form:

(a) $\log_b a + 2 \log_b c - \frac{1}{2} \log_b d$.
(b) $\frac{1}{2} \log_b a - 3 \log_b c - 4 \log_b (a + c)$.
(c) $\frac{1}{2} \log_b (a + c) + \frac{1}{2} \log_b (a - c)$.
(d) $\log_b 3c - \frac{4}{3} \log_b d + \log_b e$.
(e) $\frac{1}{3}[\log_b a + 2 \log_b (c - d) - 4 \log_b c - \frac{1}{3} \log_b (2 - a)]$.
(f) $5[\frac{1}{2} \log_b (a - c) + \log_b (a + d) - 6 \log_b d - 2 \log_b a]$.

4. Take from a four-place table the following logarithms:

$$\log_{10} 2 = 0.3010, \qquad \log_{10} 3 = 0.4771, \qquad \log_{10} 7 = 0.8451.$$

From these numbers find $\log_{10} 4$, $\log_{10} 9$, $\log_{10} 28$, $\log_{10} 32$, $\log_{10} \frac{4}{3}$, $\log_{10} \frac{3}{4}$.

5. Using the logarithms in Exercise 4, find $\log_{10} \frac{2}{3}$, $\log_{10} \frac{3}{2}$, $\log_{10} 343$, $\log_{10} \sqrt{2}$, $\log_{10} \sqrt[3]{7}$, $\log_{10} 5$.

6. Using the logarithms in Exercise 4, find the value of the logarithm of each of the following expressions:

(a) $\dfrac{(2)(5)}{3}$. (b) $\dfrac{(10)(6)}{7}$.

(c) $\dfrac{(3)(9)(5)}{14}$. (d) $\sqrt{\dfrac{(30)(21)}{8}}$.

(e) $\sqrt[5]{\dfrac{(6)(4)(7)^{\frac{1}{2}}}{28}}$. (f) $\dfrac{(9)^{\frac{1}{2}}(12)(4)^{\frac{1}{3}}}{35}$.

13-5. Common logarithms. Characteristic. In computation, it is convenient and customary to employ logarithms to the

base 10. Logarithms to this base are called **common logarithms.** Throughout this text we shall use common logarithms only, and we shall write log N as an abbreviation of $\log_{10} N$. Thus when the base is omitted it will be understood that the base is 10.

In this system of logarithms, the logarithm of any integral power of 10 is an integer, while the logarithm of any positive number not an integral power of 10 may be written as an integer plus a decimal. In general, the logarithm of a number consists of two parts, an integer called the **characteristic,** and a decimal called the **mantissa.** The characteristic is found by inspection; the mantissa is found from a table. We shall now deduce rules for finding the characteristic.

Consider the following table:

$$
\begin{array}{llll}
10^5 = 100{,}000 & \text{or} & \log 100{,}000 = & 5, \\
10^4 = 10{,}000 & \text{or} & \log 10{,}000 = & 4, \\
10^3 = 1000 & \text{or} & \log 1000 = & 3, \\
10^2 = 100 & \text{or} & \log 100 = & 2, \\
10^1 = 10 & \text{or} & \log 10 = & 1, \\
10^0 = 1 & \text{or} & \log 1 = & 0, \\
10^{-1} = 0.1 & \text{or} & \log 0.1 = & -1, \\
10^{-2} = 0.01 & \text{or} & \log 0.01 = & -2, \\
10^{-3} = 0.001 & \text{or} & \log 0.001 = & -3, \\
10^{-4} = 0.0001 & \text{or} & \log 0.0001 = & -4, \\
10^{-5} = 0.00001 & \text{or} & \log 0.00001 = & -5,
\end{array}
$$

From the foregoing table, we get by inspection the following information:

Number	Number of digits to left of decimal point	Logarithm	Characteristic
$1 < N < 10$	1	0 + a decimal	0
$10 < N < 100$	2	1 + a decimal	1
$100 < N < 1000$	3	2 + a decimal	2
$1000 < N < 10{,}000$	4	3 + a decimal	3
$10^n < N < 10^{n+1}$	$n + 1$	n + a decimal	n

From the data just tabulated, we infer the following rule:

Rule 1. The characteristic of the common logarithm of a number greater than 1 is positive and is one less than the number of digits to the left of the decimal point.

Similarly, we get

Number	Number of zeros to right of decimal point	Logarithm	Characteristic
$0.1 < N < 1$	0	$-1 +$ a decimal	-1 or $9 - 10$
$0.01 < N < 0.1$	1	$-2 +$ a decimal	-2 or $8 - 10$
$0.001 < N < 0.01$	2	$-3 +$ a decimal	-3 or $7 - 10$
$10^{-n} < N < 10^{-(n-1)}$	$n + 1$	$-n +$ a decimal	$-n$ or $(10 - n) - 10$

From the tabulated data, we infer the following rule:

Rule 2. The characteristic of the common logarithm of a positive number less than 1 is negative and is numerically one greater than the number of zeros immediately following the decimal point.

When the characteristic is negative, it is convenient to add 10 to the characteristic and subtract 10 at the right of the mantissa. Thus $\log 0.02545 = -2 + \text{a decimal} = 8 + \text{a decimal} - 10$. In general, if the characteristic $-n$ of $\log N$ is negative, change it to the equivalent value $(10 - n) - 10$, or $(20 - n) - 20$, etc. **To obtain directly the characteristic of the logarithm of a number less than 1, subtract from 9 the number of zeros immediately following the decimal point; write the result before the mantissa and -10 after it.**

Illustrations:

Number	Characteristic	Rule
4261	3	1
3.6121	0	1
0.1210	-1 or $9 - 10$	2
0.0025	-3 or $7 - 10$	2
0.00000345	-6 or $4 - 10$	2

EXERCISES 13-4

Write the characteristic of the logarithm of each number:

1. 7.613.	**2.** 467,916.	**3.** 20.02.	**4.** 3.00008.
5. 761.3.	**6.** $31.1\dot{2}$.	**7.** 0.0371.	**8.** 0.81219.
9. 89,261.	**10.** 412.16.	**11.** 0.0000309.	**12.** 0.003872.
13. 3101.	**14.** 14,481.10.	**15.** 0.30001.	**16.** 0.000810.

13-6. Effect of changing the decimal point in a number. Any number may be written in the form $N \times 10^k$, where N is a number between 1 and 10 and k is an integer. Thus we may write $1{,}782{,}500 = 1.7825 \times 10^6$, $17825 = 1.7825 = 10^4$. Evidently a shift of the decimal point appears in this notation as a change in k. Now $\log [N \times 10^k] = \log N + k \times 1$. Since a shift of the decimal point changes k, but not $\log N$, it appears that the mantissa of $\log N$ is not affected by the position of the decimal point. In other words, a change in the position of the decimal point in a given sequence of figures has no effect on the mantissa; its sole effect is to change the characteristic. Because of this fact, 10 affords a particularly convenient base for a system of logarithms to be used for purposes of computation.

13-7. The mantissa. Mantissas can be computed by use of advanced mathematics and, except in special cases, are unending decimal fractions. Computed mantissas are tabulated in tables of logarithms, also called tables of mantissas. These tables are called "three-place," "four-place," "five-place," etc., according as the mantissas tabulated contain 3, 4, 5, etc., significant figures. The choice of a table of logarithms should depend upon the degree of accuracy required and the accuracy of the data. In this text we shall discuss and use a four-place table, thus obtaining results accurate to four significant figures.

13-8. To find the logarithm of a number. The first two digits of the numbers are found in the left-hand column headed N, and the third digit is in the row at the top of the page. Therefore the mantissa of a number with three significant figures is in the row with the first two significant figures of the number and in the column headed by the third.

Example 1. Find log 42.4.

Solution. By the rule in Art. 13-5, the characteristic is found to be 1. To find the mantissa, first find 42 in the left-hand column headed N, then follow the row containing 42 until the column headed by 3 is reached. Here we find 6274. Therefore the mantissa is 0.6274. Hence

$$\log 42.43 = \mathbf{1.6274}.$$

Example 2. Find log 0.0416.

Solution. By the rule in Art. 13-5, the characteristic is found to be 8. -10. Using 416, we find the mantissa to be 0.6191. Therefore

$$\log 0.0416 = \mathbf{8.6191 - 10}.$$

EXERCISES 13-5

Verify the following.

1. $\log 293 = 2.4669$.
2. $\log 3.47 = 0.5403$.
3. $\log 28.7 = 1.4579$.
4. $\log 1.82 = 0.2601$.
5. $\log 981 = 2.9917$.
6. $\log 0.313 = 9.4955 - 10$.
7. $\log 0.000314 = 6.4969 - 10$.
8. $\log 0.0342 = 8.5340 - 10$.
9. $\log 0.272 = 9.4346 - 10$.
10. $\log 0.00507 = 7.7050 - 10$.

13-9. Interpolation. From the four-place table of logarithms we cannot obtain directly the logarithm of a number with three significant figures. However, by a process known as interpolation, we can find the mantissa of a number having a fourth significant figure. In this process we use the principle of proportional parts, which states that, for small changes in N, the corresponding changes in $\log N$ are proportional to the changes in N. Although this principle is not strictly true, it is sufficiently accurate to lead to results correct to the number of figures given in the table.

The process of interpolation is illustrated by means of the following example:

Example. Find log 235.4.

Solution. From the table of logarithms we find the logarithms in the following form and then compute the differences exhibited.

$$\left.\begin{array}{l}\left.\begin{array}{l}\log 235.0\\ \log 235.4\end{array}\right\} 4\\ \log 236.0\end{array}\right\} 10 \quad \left.\begin{array}{l}\left.\begin{array}{l}= 2.3711\\ = \ ?\end{array}\right\} d\\ = 2.3729\end{array}\right\} 0.0018 \text{ (tabular difference)}$$

By the principle of proportional parts, we have

$$\frac{4}{10} = \frac{d}{0.0018}, \qquad \text{or} \qquad d = \frac{4}{10}(0.0018) = 0.00072.$$

We add 0.0007 to 2.3711 to obtain log 235.4 = **2.3718.**

Notice that the value of d was 0.0007 instead of 0.00072 because the table of logarithms is accurate to four decimal places.

EXERCISES 13-6

Find the logarithm of each of the following:

1. 40.48. **2.** 3.047.
3. 1029. **4.** 108.1.
5. 0.2154. **6.** 0.003834.
7. 0.08645. **8.** 0.00007612.
9. 0.02703. **10.** 0.1825.

13-10. To find the number corresponding to a given logarithm. Generally in every problem involving logarithms, it is necessary not only to find the logarithms of numbers but also to perform the inverse process, that of finding a number corresponding to a given logarithm.

If $\log N = L$, then N is the number corresponding to the logarithm L. The number N is called the *antilogarithm* of L. To find the antilogarithm N of the logarithm L, first use the given mantissa to find the sequence of figures in N, and then use the given characteristic to place the decimal point so as to agree with the rule of Art. 13-5.

Example. Given $\log N = 1.6033$. Find N.

Solution. The mantissa .6033 is not found exactly in the table, but we find the two successive mantissas .6031 and .6042 between which the given mantissa lies. From the table we find the numbers in the following form and then compute the differences exhibited.

$$\left.\begin{array}{l}\left.\begin{array}{l}1.6031\\1.6033\end{array}\right\}0.0002\\1.6042\end{array}\right\}0.0011\quad\left.\begin{array}{l}\left.\begin{array}{l}=\log 40.10\\=\log N\end{array}\right\}x\\=\log 40.20\end{array}\right\}10$$

By the principle of proportional parts, we have

$$\frac{x}{10} = \frac{.0002}{.0011}, \qquad \text{or} \qquad x = \frac{(10)(.0002)}{.0011} = 2 \text{ (nearly)}.$$

We add the 2 to the last figure of 40.10 to obtain N = **40.12.**

EXERCISES 13-7

Find x in each of the following:

1. $\log x = 8.6630 - 10.$
2. $\log x = 3.8977.$
3. $\log x = 2.3166.$
4. $\log x = 9.7000 - 10.$
5. $\log x = 7.9729 - 10.$
6. $\log x = 2.9987.$
7. $\log x = 0.8748.$
8. $\log x = 0.4223.$
9. $\log x = 1.1124.$
10. $\log x = 6.5474 - 10.$

13-11. The use of logarithms in computations. The following examples will illustrate how logarithms are used.

Example 1. Evaluate (461)(4.321).
Solution. Denoting the product by x, we may write

$$x = (461)(4.321).$$

Equating the logarithms of the two members of this equation, we get

$$\log x = \log 461 + \log 4.321.$$

Looking up the logarithms of the numbers, we obtain

$$\begin{aligned} \log 461 &= 2.6637 \\ \log 4.321 &= 0.6356 \end{aligned}$$

Adding, we have $\log x = 3.2993$
Therefore, the antilogarithm x = **1922.**

Example 2. Evaluate $\dfrac{(217)(3.18)}{62.14}$.

Solution. Let $x = \dfrac{(217)(3.18)}{62.14}$.

Then $\log x = \log 217 + \log 3.18 - \log 62.14.$

$$\begin{aligned}
\log 217 &= 2.3365\\
\log 3.18 &= 0.5024\\
\text{Sum} &= 2.8389\\
\log 62.14 &= 1.7934
\end{aligned}$$

Subtracting, we obtain $\log x = 1.0455$

The antilogarithm $x = \mathbf{11.11.}$

Example 3. Evaluate $(2.713)^3$.

Solution. Let $x = (2.713)^3$. Then

$$\log x = 3 \log 2.713 = 3(0.4335) = 1.3005.$$
$$\therefore x = \mathbf{19.98.}$$

Example 4. Evaluate $\sqrt[3]{0.7214}$.

Solution. Let $x = \sqrt[3]{0.7214} = (0.7214)^{\frac{1}{3}}$. Then

$$\log x = \tfrac{1}{3} \log 0.7214 = \tfrac{1}{3}(9.8581 - 10).$$

If we should divide this logarithm by 3, the characteristic of the resulting logarithm would not be in the standard form. Hence we first add 20 and then subtract 20, writing the logarithm in the form 29.8581 − 30. Then we write

$$3)\underline{29.8581 - 30}$$

Dividing, we get $\log x = 9.9527 - 10$

or $x = \mathbf{0.8968.}$

EXERCISES 13-8

Evaluate the following:

1. 5256×0.008254. **2.** $37.92 \div 5.3$. **3.** $(1.045)^{12}$.

4. $(0.03628)^{\frac{1}{4}}$. **5.** $\sqrt[3]{(442.8)^2}$. **6.** $(33.98)^{\frac{2}{3}}$.

7. $\dfrac{0.003159 \times 684.8}{0.009654}$. **8.** $\dfrac{7585 \times 0.002824}{3756 \times 0.09185}$.

13-12. Cologarithms. Subtracting a first number from a second is equivalent to adding the negative of the first to the second. Hence, to avoid subtraction in dealing with logarithms, we introduce cologarithms.

The cologarithm of a number is the negative of its logarithm. Therefore, adding the cologarithm of a number is equivalent to subtracting its logarithm.

To avoid negative mantissas, the cologarithm of a number n, written colog n, is found by using the form

$$\operatorname{colog} n = 10 - \log n - 10.$$

Thus

$$\operatorname{colog} 2 = 10 - \log 2 - 10 = 10 - 0.3010 - 10 = 9.6990 - 10,$$

and colog $0.3 = 10 - (9.4771 - 10) - 10 = 0.5229$. The student will find it convenient in getting colog n to *begin at the left of log n, subtract each of its digits from* 9 *except the last significant one, and subtract that from* 10.

The following example will illustrate the use of cologarithms.

Example. Find x if $x = \dfrac{342.1}{(6710)(0.3182)}$.

Solution.

$$\begin{aligned} \log x &= \log 342.1 - \log 6710 - \log 0.3182 \\ &= \log 342.1 + \operatorname{colog} 6710 + \operatorname{colog} 0.3182 \end{aligned}$$

$$\begin{array}{ll} & \log 342.1 = 2.5341 \\ \log 6710 \quad = 3.8267, & \operatorname{colog} 6710 = 6.1733 - 10 \\ \log 0.31820 = 9.5027 - 10, & \operatorname{colog} 0.3182 = 0.4973 \\ & \log x = 9.2047 - 10 \end{array}$$

and $x =$ **0.1602.**

EXERCISES 13-9

1. Verify the following:

(*a*) colog $179.8 = 7.7452 - 10$.
(*b*) colog $0.6327 = 0.1988$.
(*c*) colog $7.532 = 9.1231 - 10$.
(*d*) colog $23.97 = 8.6203 - 10$.

2. Using cologarithms, find the value of

(*a*) $\dfrac{36.21}{7.215}$. (*b*) $\dfrac{42.21}{0.2861}$. (*c*) $\dfrac{41.26}{(61.84)(1612)}$. (*d*) $\dfrac{142.3}{0.02813}$.

13-13. Computation by logarithms. In solving complicated problems, the computer is helped materially by a good form. The one discussed below has the advantages of simplicity, completeness of record, and brevity. It is practically self-

explanatory since the main feature consists in reference of every function on a line to the first number in the line; a complete record of logarithms and operations is tabulated, and little writing is required. Since the outline of the form can always be made in advance, the student should first make this outline and then perform the computation without interruption. Speed and accuracy are gained by this method.

The form will be used in the following solution:

Example 1. Find x if $x = \dfrac{a^{\frac{1}{3}} \sqrt[5]{b}\, c^2}{de^4}$ and $a = 8.163$, $b = 729.7$, $c = 0.0463$, $d = 5.213$, $e = 0.3241$.

Solution. First write the formula

$$\log x = \tfrac{1}{3} \log a + \tfrac{1}{5} \log b + 2 \log c + \text{colog } d + 4 \text{ colog } e.$$

The following form contains the solution:

$a = 8.163$	$\log a = 0.9119$	$\frac{1}{3} \log a = 0.3040$
$b = 729.7$	$\log b = 2.8631$	$\frac{1}{5} \log b = 0.5726$
$c = 0.0463$	$\log c = 8.6656 - 10$	$2 \log c = 7.3312 - 10$
$d = 5.213$	$\log d = 0.7171$	$\text{colog } d = 9.2829 - 10$
$e = 0.3241$	$\log e = 9.5016 - 10$	$4 \text{ colog } e = 1.9576$
$x = \mathbf{0.2807}$		$\log x = 9.4483 - 10$

In the following solution a form is indicated, but the computation is left as in exercise to the student.

Example 2. Find x if $x = \left[\dfrac{\sqrt{c} \times a^2}{a + \sqrt{e}}\right]^{\frac{1}{3}}$ where $a = 61.21$, $c = 12.11$, and $e = 139.1$.

Solution. First we write the formula

$$\log x = \tfrac{1}{3}[\tfrac{1}{2} \log c + 2 \log a + \text{colog } (a + \sqrt{e})]$$

and then make the following form:

$e = 139.1$	$\log e =$	$\frac{1}{2} \log e =$	
$\sqrt{e} =$		$\log \sqrt{e} =$	
$a = 61.21$	$\log a =$		$2 \log a =$
$a + \sqrt{e} =$	$\log (a + \sqrt{e}) =$		$\text{colog } (a + \sqrt{e}) =$
$e = 12.11$	$\log c =$		$\frac{1}{2} \log c$
			3)
$x =$			$\log x =$

The student should perform the computation to obtain

$$x = \mathbf{5.633}.$$

EXERCISES 13-10

Make a form or outline for computing each of the following:

1. $\dfrac{(32.86)^2(3.141)^{\frac{1}{3}}}{(62.18)^3}$.

2. $\sqrt[3]{\dfrac{(31.64)^2(62.12)}{(9.31)^5}}$.

3. $\left[\dfrac{a^2b^3c^{\frac{1}{2}}}{d^5e}\right]^2$.

4. $\sqrt[5]{\dfrac{a^2\sqrt{b}\sqrt[3]{c}}{d^3\sqrt{e}}}$.

13-14. Remarks on computation by logarithms.

(*a*) *When interpolating, do not carry logarithms beyond the number of decimal places given in the table used.*

(*b*) *When evaluating an expression containing negative numbers, use logarithms to compute the desired positive components, and then combine the results with appropriate signs. In this text a symbol* $(-)$ *before a logarithm will indicate that a negative number is under consideration; thus if* log $x = (-)9.87123 - 10$, $x = -0.74342$.*

(*c*) *Make a form like that of Example* 1, *Art.* 13-13, *before beginning computation.*

(*d*) *Strive for accuracy in computation. Speed comes with* practice.

Example. Find the value of x if $x = \sqrt[5]{\dfrac{(-47.12)^2(-36.18)^{\frac{1}{3}}}{\sqrt{31.11}}}$.

Solution.

$$\log(-x) = \tfrac{1}{5}[2 \log 47.12 + \tfrac{1}{3} \log 36.18 + \tfrac{1}{2} \text{colog } 31.11].$$

$a = -47.12$	$\log a = (-)1.6732$	$2 \log a = 3.3464$
$b = -36.18$	$\log b = (-)1.5585$	$\frac{1}{3}\log b = (-)0.5195$
$c = 31.11$	$\log c = 1.4929$	$\frac{1}{2}\text{colog } c = 9.2535 - 10$
		5) $(-)3.1194$
$x = \mathbf{-4.206}$		$\log x = (-)0.6239$

EXERCISES 13-11

Find by use of logarithms the results of the following exercises. In each case make a complete outline or form before using the tables.

* This does not mean that a negative number has a real logarithm. The minus symbols serve merely to keep a record of the signs involved in the given expression.

1. 3.142×2.718.

2. $\sqrt{347.3}$.

3. 29.57×0.00368.

4. $(1.5)^5$.

5. $1487 \times 3.139 \times 42.96$.

6. $\sqrt[3]{31}$.

7. $272.7 \div 37.37$.

8. $\sqrt[3]{0.1764 \times 2.128}$.

9. $(0.0006258)^{\frac{1}{8}}$.

10. $\sqrt{(27.5)^2 - (3.483)^2}$.

11. $\dfrac{2.928 \times 34.27}{505.9}$.

12. $\dfrac{48.96 \times 39.59}{78.55}$.

13. $\dfrac{296.4 \times 38.42}{75.65 \times 84.38}$.

14. $\dfrac{295.4 \times 64.53}{911.3 \times 318.5}$.

15. $\left[\dfrac{198.7}{38.34}\right]^2$.

16. $\sqrt{\dfrac{57.45 \times 423.3}{178 \times 89}}$.

17. $\dfrac{(-8094)\sqrt[5]{-0.031}}{5408\sqrt[6]{0.0712}}$.

18. $\dfrac{4 \times 28.7 \times \sqrt{345}}{29 \times 137}$.

19. $\sqrt[3]{\dfrac{a^{\frac{1}{3}}b}{a^2 - b}}$, $a = 7.532$, $b = 6384$.

20. $\sqrt[5]{\dfrac{b}{a^3}} - \sqrt{a^2c}$; $a = 735.9$, $b = 0.198$, $c = 27$.

21. $\dfrac{a^2c^{\frac{1}{2}}}{bD}$; $D = a + c^2$, $a = 23.72$, $b = 571.1$, $c = 0.0321$.

22. Given $a = 3.712$, $b = 32.61$, find $\log(a + b)$, $\log(a - b)$, $\log\dfrac{a}{b}$, $\log ab$.

23. Find K, given $s = \frac{1}{2}(a + b + c + d)$,

$$K = \sqrt{(s - a)(s - b)(s - c)(s - d)},$$

$a = 6.324$, $b = 7.745$, $c = 8.544$, $d = 5.196$.

24. $\dfrac{a^3b^2c}{d^{\frac{1}{3}}}$, given $a = 0.00275$, $b = 100.5$, $c = 507.6$, $d = 0.001875$.

25. $\left[\dfrac{a^5b^3c^2d^{\frac{1}{5}}}{e^2f^3g^4}\right]^{\frac{1}{8}}$, given $a = 301.1$, $b = 0.0003695$, $c = 0.002818$, $d = 35{,}890{,}000$, $e = 0.000002814$, $f = 561.2$, $g = 2718.3$.

26. Find the weight of a steel sphere 1.012 ft. in diameter if steel weighs 490 lb. per cu. ft.

27. Find the weight of a cube of metal weighing 530 lb. per cu. ft. if the edge of the cube is 1.627 ft.

28. A conical piece of wood weighs 92 lb. If the area of the base of the solid is 1.334 sq. ft., find the altitude. (The wood weighs 33 lb. per cu. ft.)

29. During a rain 0.521 in. of water fell. Find how many gallons of water fell on a level 10.7-acre park. (Take 1 cu. ft. = 7.48 gal., 1 acre = 43,560 sq. ft.)

30. The time t of oscillation of a simple pendulum of length l ft. is given in seconds by the formula

$$t = \pi \sqrt{\frac{l}{32.16}}.$$

Find the time of oscillation of a pendulum 3.326 ft. long. (Take $\pi = 3.142$.)

31. What is the weight in tons of a solid cast-iron sphere whose radius is 5.343 ft. if the weight of 1 cu. ft. of water is 62.36 lb. and the specific gravity of cast iron is 7.154?

32. Find the volume and surface of a sphere of radius 14.71.

33. The stretch of a brass wire when a weight is hung at its free end is given by the relation

$$S = \frac{mgl}{\pi r^2 k},$$

where m is the weight applied, $g = 980$, l is the length of the wire, r is its radius, and k is a constant. Find k for the following values: $m = 944.2$ g., $l = 219.2$ cm., $r = 0.32$ cm., and $S = 0.060$ cm.

34. Find the length l of a wire that stretches 5.9 cm. for a weight of 1826 g. hanging at its free end, when the diameter of the wire is 0.064 cm. and $k = 1.1 \times 10^{12}$.

35. The weight P in pounds that will crush a solid cylindrical cast-iron column is given by the formula

$$P = 98{,}920 \frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches and l the length in feet. What weight will crush a cast-iron column 6 ft. long and 4.3 in. in diameter?

36. For wrought-iron columns the crushing weight is given by

$$P = 299{,}600 \frac{d^{3.55}}{l^2}.$$

What weight will crush a wrought-iron column of the same dimensions as that in Problem 35?

37. The weight W of 1 cu. ft. of saturated steam depends upon the pressure in the boiler according to the formula

$$W = \frac{P^{0.941}}{330.4},$$

where P is the pressure in pounds per square inch. What is W if the pressure is 280 lb. per sq. in.?

13-15. Change of base in logarithms. Occasionally it is necessary to find the logarithm of a number N to a base b other than 10. To do this we let

$$\log_b N = x, \qquad \text{or} \qquad b^x = N.$$

Equating the logarithms to the base 10 of the two members of this equation, we get

$$x \log_{10} b = \log_{10} N, \qquad \text{or} \qquad x = \frac{\log_{10} N}{\log_{10} b}.$$

Since the divisor and dividend of this fraction are logarithms, they will generally be numbers of several digits. Therefore it is advisable to perform the indicated division by means of logarithms.

Example. Find the value of $\log_3 0.09211$.

Solution. Let $x = \log_3 0.09211$. Then $3^x = 0.09211$. Equating the logarithms to the base 10 of the two members of this equation, we obtain

$$x \log_{10} 3 = \log_{10} 0.09211$$

or

$$x = \frac{\log_{10} 0.09211}{\log_{10} 3} = \frac{8.9643 - 10}{0.4771} = \frac{-1.0357}{0.4771}.$$

This quotient is evaluated as follows:

$a = -1.0357$		$\log a = (-)0.0152$
$b = 0.4771$	$\log b = 9.6786 - 10$	$\text{colog } b = 0.3214$
$x = \mathbf{-2.171}$		$\log x = (-)0.3366$

13-16. Solution of equations of the form $x = a^b$, $a = x^b$. We shall now illustrate the method of solving equations of the form $x = a^b$, and $a = x^b$, in which a and b are given numbers.

Example 1. Find x if $x = (3.21)^{8.27}$.

Solution. $\log x = 8.27 \log 3.21 = (8.27)(0.5065)$.

The solution is displayed below.

$a = 8.27$	$\log a = 0.9175$
$b = 0.5065$	$\log b = 9.7046 - 10$
$\log x = 4.1880$	$\log (\log x) = 0.6221$

$\therefore \log x = 4.1889$ from which we get $x = \mathbf{15{,}450}$.

Example 2. Find x if $x^{7.214} = 0.08013$.

Solution. Equate the logarithms of the two members of the given equation and solve for $\log x$ to obtain

$$7.214 \log x = \log 0.08013$$

or

$$\log x = \frac{\log 0.08013}{7.214} = \frac{8.9038 - 10}{7.214} = \frac{-1.0962}{7.214}.$$

The evaluation of the quotient for $\log x$ follows:

$$\begin{array}{rrl|lcl} a = & -1.0962 & & \log a & = & (-)0.0399 \\ b = & 7.214 & \log b = 0.8581 & \operatorname{colog} b & = & 9.1419 - 10 \\ \hline \log x = & -0.1520 & & \log(\log x) & = & (-)9.1818 - 10 \end{array}$$

To make the mantissa of $\log x$ positive add it to $10 - 10$ to obtain

$$\log x = 10 - 0.1520 - 10 = 9.8480 - 10.$$
$$\therefore x = \mathbf{0.7047}.$$

EXERCISES 13-12

1. $x = \log_7 100$.

2. $x = \log_{0.88} 9{,}932$.

3. $x = \log_{27} 0.00328$.

4. $x = \log_{0.0954} 87.54$.

5. $x = \log_{20} 100$.

6. $x = \log_8 2{,}756$.

7. $x = \log_{3.7} 0.8173$.

8. $x = \log_{21} 0.09827$.

9. $5^{\frac{1}{x}} = 1.307$.

10. $5^{2x} = 317.4$.

11. $\log_x 8 = 0.3567$.

12. $\log_x 2 = 0.6931$.

13. $\log_x 0.07936 = 2.983$.

14. $x^{2.892} = 0.07936$.

15. $(1.5)^{\frac{1}{x}} = 32$.

16. $4.02 = (2.37)^{\frac{1}{x+1}}$.

17. Given $3^{x+y} = 2(5^x)$, $x - y = 1$, find x and y.

18. How long will it take a sum of money to double itself if put at 4 per cent compound interest? This is represented by $(1.04)^x = 2$ where x is the number of years. Solve for x.

19. Solve the equation $e^x + e^{-x} = y$, for x (*a*) when $y = 2$, (*b*) when $y = 4$. $e = 2.718$.

20. If fluid friction is used to retard the motion of a flywheel making V_0 revolutions per minute, the formula $V = V_0e^{-kt}$ gives the number of revolutions per minute after the friction has been applied t sec. If the

constant $k = 0.35$, how long must the friction be applied to reduce the number of revolutions from 200 to 50 per minute? $e = 2.718$.

21. The pressure, P, of the atmosphere in pounds per square inch, at a height of z ft. is given approximately by the relation

$$P = P_0 e^{-kz},$$

where P_0 is the pressure at sea level and k is a constant. Observations at sea level give $P_0 = 14.72$, and at a height of 1122 ft., $P = 14.11$. What is the value of k?

22. Assuming the law in Exercise 21 to hold, at what height will the pressure be half as great as at sea level?

23. If a body of temperature $T_1°$ is surrounded by cooler air of temperature $T_0°$, the body will gradually become cooler, and its temperature, $T°$, after a certain time, say t min., is given by Newton's law of cooling, that is,

$$T = T_0 + (T_1 - T_0)e^{-kt},$$

where k is a constant. In an experiment a body of temperature 55°C. was left to itself in air whose temperature was 15°C. After 11 min. the temperature was found to be 25°. What is the value of k?

24. Assuming the value of k found in Exercise 23, what time will elapse before the temperature of the body drops from 25° to 30°?

25. Solve the equation $\log_e (3x + 1) = 2$ for x.

26. Solve the equation $\log_{10} (x^2 - 21x) = 2$ for x.

13-17. Graph of $y = \log_{10} x$. If we assign values to x in the equation $y = \log_{10} x$ and find the corresponding values of y, we shall obtain the coordinates of points on the curve $y = \log_{10} x$. A few of these values are tabulated in the accompanying table. Plotting these points and drawing a smooth curve through

x	0.5	1	3	5	8	10	15	20	25	30	35	40
y	−0.3	0	0.48	0.70	0.9	1	1.17	1.3	1.4	1.48	1.54	1.6

them, we obtain the graph shown in Fig. 13-1. For convenience, the unit on the y-axis has been taken ten times as large as the unit on the x-axis.

If the student retains a mental picture of this graph, he will find it easy to recall the following facts:

(*a*) A negative number has no real number for its logarithm.

(*b*) The logarithm of a positive number is negative or positive according as the number is less than or greater than 1.

(*c*) If the number x approaches zero, log x decreases without limit.

(*d*) If the number x increases indefinitely, log x increases without limit.

In the process of interpolation in logarithms, values are inserted as if the change in the logarithm between the nearest tabulated values were directly proportional to the change in the number. This assumes that the graph of $y = \log x$ for the interval concerned is a straight line. From the graph it is apparent this would be approximately true. In other words, when a number is changed by an amount that is very small in comparison with the number itself, the change in the value of the logarithm of the number is very nearly proportional to the change in the number.

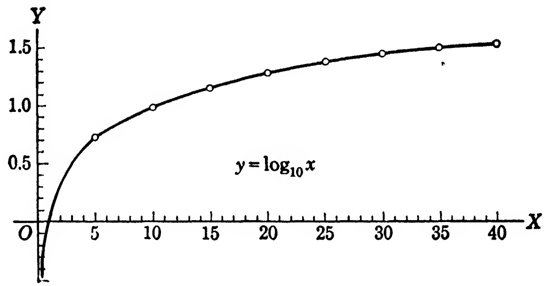


FIG. 13-1.

EXERCISES 13-13

1. Plot the graph of $y = \log_5 x$.

Hint. $\log_5 x = \dfrac{\log_{10} x}{\log_{10} 5}$.

2. Plot the graph of $x = \log_5 y$.

3. Plot the graph of $x = \log_2 y$.

MISCELLANEOUS EXERCISES 13-14

Find by the use of logarithms the results of the following exercises. In each case make a complete outline or form before using the tables.

1. $3.87 \times 57.6.$

2. $7.092 \times 0.005268.$

3. $22.9 \times 4.95 \times 0.643.$

4. $0.006398 \times 23.47 \times 0.06254.$

5. $\dfrac{76.9}{3.14}.$

6. $\dfrac{1}{0.8236}.$

7. $\dfrac{8.211}{0.6634}.$

8. $\dfrac{49.36 \times 0.7657}{8.439}.$

9. $\dfrac{6.47 \times 12.93 \times 0.2462}{896 \times 0.007493}$

10. $(0.09245)^3.$

11. $\sqrt[6]{0.002855}$

12. $\sqrt[4]{0.007001}.$

13. $(0.935)^{\frac{3}{5}}.$

14. $(4.267)^{0.4}.$

15. $(19.26)^{1.2}.$

16. $\dfrac{(41.91)^{\frac{5}{4}}}{\sqrt[5]{(3.215)^3} \times 0.7835}.$

17. $\dfrac{(89.1)^{\frac{2}{3}} \times (0.764)^{0.2}}{\sqrt[4]{0.0387}}$

18. $\dfrac{(7.903)^{1.1} \times \sqrt[5]{(0.5026)^3}}{(0.001412)^{0.9}}.$

19. $(-0.09111)^{-\frac{3}{5}}.$

20. $\dfrac{45.86 \times (0.7288)^{\frac{3}{4}}}{(-9.423)^{\frac{5}{3}}}.$

21. $\dfrac{(-0.04917)^{\frac{2}{3}}}{\sqrt[5]{-207.9}}.$

22. $\dfrac{1}{\sqrt[5]{(170.5)^3} - 15}.$

23. $\dfrac{\sqrt{0.7285} + (2.706)^{\frac{3}{2}}}{318.2 \times (0.06004)^2}.$

24. $\dfrac{(0.8195)^{-0.3} + (0.9713)^{0.4}}{(5.004)^{-\frac{1}{3}}}.$

25. $\dfrac{\log 9.5}{\log 4.27}.$

26. $\dfrac{\log 0.8718}{\log 0.02222}.$

27. The radius r of the inscribed circle of a triangle in terms of its sides a, b, and c is given by

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

where $s = \frac{1}{2}(a + b + c)$. Compute r when (*a*) $a = 0.525$, $b = 0.261$, $c = 0.438$; (*b*) $a = 698.2$, $b = 476.3$, $c = 744.9$; (*c*) $a = 3.002$, $b = 2.113$, $c = 1.501$.

28. The number n of revolutions per minute of a certain water turbine is given by

$$n = \frac{400}{61.3} h^{1.3} P^{-0.4},$$

29. The formula $D = \sqrt[3]{\dfrac{W}{0.5236(A - G)}}$ gives the diameter of a spherical balloon which is to lift a cable of weight W. Find D if $A = 0.0807$, $G = 0.0050$, $W = 1250$.

30. The amount S of a principal of P dollars, interest compounded annually for n years at the rate i, is

$$S = P(1 + i)^n.$$

If a war bond sells today for \$75 and will be redeemed in 10 years for \$100, what rate of interest compounded annually will be paid?

Hint. $S = 100$, $P = 75$, $n = 10$.

31. The range R on a horizontal plane of a projectile fired at an angle θ, with velocity v_0, is

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

Find the muzzle velocity of a projectile fired at sea whose maximum range is 22.7 miles.

Hint. $R = 22.7 \times 6080$ ft., $g = 32.17$ ft. per sec. per sec., $\theta = 45°$.

32. If the height y in feet of a projectile above a horizontal plane at time t in seconds is given by the equation

$$y = -16t^2 + 600t,$$

show that its height at $t = 18.75$ sec. is 5625 ft.

33. If the height y (see Fig. 13-2) of a projectile in terms of the horizontal distance x from the gun is given by

$$y = x \tan \theta - \frac{\frac{1}{2}gx^2}{v_0^2 \cos^2 \theta},$$

where θ is the angle of elevation of the gun, v_0 is the initial velocity, and

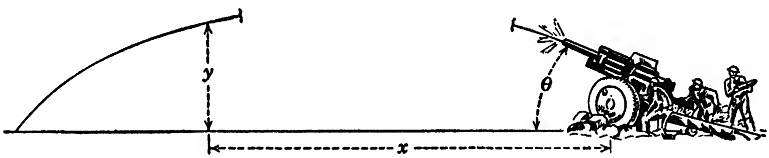


FIG. 13-2.

$g = 32$ ft. per sec. per sec. (approx.), find y when $x \doteq 38{,}970$ ft., $\theta = 30°$, $v_0 = 2400$ ft. per sec.

34. The expressions

$$x = 104.6t$$
$$y \doteq 6070(1 - e^{-0.0322t}) + 1000t$$

give the horizontal distance x and the vertical distance y at time t of a shell projected from an airplane at an angle of 85° below the horizontal,

with an initial velocity of 1200 ft. per sec. Find the position of the shell at the end of 5 sec. (see Fig. 13-3).

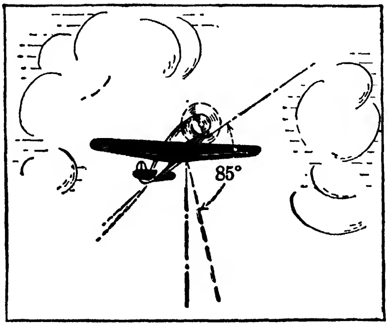


Fig. 13-3.

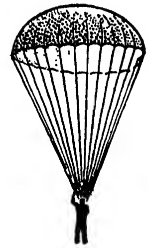


Fig. 13-4.

35. If the air pressure on the ground is 14.7 lb. per sq. in., the pressure P at height h ft. is given approximately by

$$P = 14.7e^{-0.0000377h}.$$

Find the air pressure at the height of (*a*) 10,000 ft., (*b*) 15,000 ft.

36. If the force F exerted by a parachute on a man of weight W lb. falling v ft. per sec. is given by

$$F = \frac{Wv}{15},$$

find the force exerted on a 160-lb. man by a parachute just as it opens if he is then falling at 98 ft. per sec. (see Fig. 13-4).

37. When a ship is displaced from its vertical position, it makes a complete oscillation by rolling from port to starboard and back in a time t sec. given by

$$t = 2\sqrt{\frac{r^2}{gm}},$$

where $g = 32.17$, r is a constant depending on the weight and shape of the ship, and m is the metacentric height. If $r = 38.06$ ft.,

$$m = 7.874 \text{ ft.},$$

$g = 32.17$ ft. per sec. per sec., find the time of an oscillation of the ship.

38. An airplane descending with a speed of 120 miles per hour at an angle of 20° with the horizontal drops a bomb when 700 ft. high (see Fig. 13-5). The vertical distance y and the horizontal distance x of the bomb from the point of release are given by the equations

$$y = 60.2t + 16.1t^2,$$
$$x = 165.4t.$$

(*a*) Find the distance the bomb moves horizontally if it strikes the warship shown in the figure in 4.98 sec. (*b*) Find the angle of depression θ of the target as observed by the pilot when releasing the bomb. (*c*) Find the vertical distance the bomb falls during the first 2.5 sec.

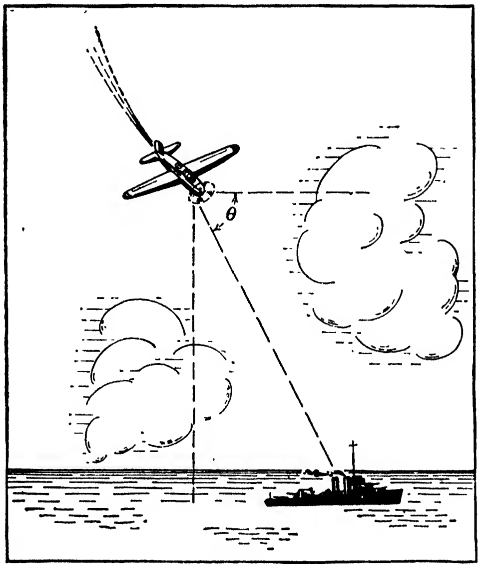


FIG. 13-5.

39. Find the total time required for a 23.8-knot torpedo to make its maximum run of 12,640 yd. Take 2027 yd. = 1 nautical mile and assume the speed as constant.

40. In a certain situation the captain of a warship desired to come as close to an enemy scout as possible. The time in hours required to attain this position is given by the formula

$$\text{Time} = \frac{bc}{a(a^2 - b^2)^{\frac{1}{2}}}.$$

where c = initial distance of the scout from the warship, a = speed in knots of the scout, b = speed in knots of warship. Find the time required if b = 28.4 knots, a = 32.7 knots, c = 20.8 nautical miles.

41. The formula $y = 0.0263x^{1.1}$ gives the relation between y and x when x stands for the stress in kilograms per square centimeter of cross section of a hollow cast-iron tube subject to tensile stress and y for

the elongation of the tube in terms of $\frac{1}{600}$ cm. as a unit. Compute y when $x = 101.8$.

42. The formula $y = ks^x g^{c^x}$, where $\log k = 5.0337$, $\log s = -0.003$, $\log g = -0.0001$, $\log c = 0.045$, gives the number living at age x in Hunter's Makehamized American Experience Table of Mortality. Find, to such a degree of accuracy as you can secure with a four-place table of logarithms, the number living (*a*) at age ten, (*b*) at age thirty.

43. Given that 1 km. = 0.6214 mile. Find the number of miles in 2489 km.

44. Given that 1 km. = 0.6214 mile and that the area of Illinois is 56,625 square miles. Express the area of Illinois in square kilometers (to four significant figures).

CHAPTER 14

THE SLIDE RULE

14-1. Introduction. This chapter, while giving a brief review of the method of using a slide rule, stresses the settings relating to trigonometry. The settings given apply to most slide rules, but the explanation is based on the manuals written by Kells, Kern, and Bland for the slide rules manufactured by the Keuffel and Esser Company. For a logarithmic explanation of this slide rule and more detail concerning the settings, the student is referred to the manuals just cited.

Efficient operation of a slide rule is a comparatively simple matter. Since nearly every setting is based on one principle called the *proportion principle*, it is easy to recall forgotten settings and devise new ones especially suited to the work at hand. The first step is to learn to read the scales on the rule.

14-2. Reading the scales.* Figure 14-1 represents, in skeleton form, the fundamental scale of the slide rule, namely the D scale.

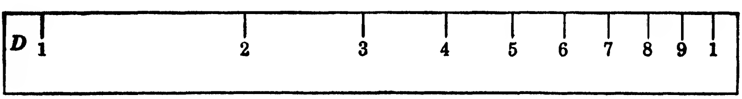


FIG. 14-1.

An examination of this actual scale on the slide rule will show that it is divided into 9 parts by primary marks that are numbered 1, 2, 3, . . . , 9, 1. The space between any two primary marks is divided into ten parts by nine secondary marks. These are not numbered on the actual scale except between the primary marks numbered 1 and 2. Figure 14-2 shows the secondary marks lying between the primary marks of the D scale. On this scale each italicized number gives the reading to be associated

* The description here given has reference to the 10-in. slide rule. However, anyone having a rule of different length will be able to understand his rule in the light of the explanation given.

with its corresponding secondary mark. Thus, the first secondary mark after 2 is numbered 21, the second 22, the third 23, etc.; the first secondary mark after 3 is numbered 31, the second 32, etc. Between the primary marks numbered 1 and 2 the secondary marks are numbered 1, 2, . . . , 9. Evidently the readings associated with these marks are 11, 12, 13, . . . , 19. Finally between the secondary marks (see Fig. 14-3) appear smaller or tertiary marks that aid in obtaining the third digit of a reading. Thus between the secondary marks numbered 22 and 23 there are four tertiary marks. If we think of the end marks as repre-

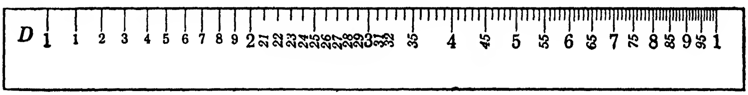


FIG. 14-2.

senting 220 and 230, the four tertiary marks divide the interval into five parts, each representing two units. Hence with these marks we associate the numbers 222, 224, 226, and 228; similarly the tertiary marks between the secondary marks numbered 32 and 33 are read 322, 324, 326, and 328, and the tertiary marks between the primary marks numbered 3 and the first succeeding secondary mark are read 302, 304, 306, and 308. Between any pair of secondary marks to the right of the primary mark numbered 4, there is only one tertiary mark. Hence, each smallest space represents five units. Thus the primary mark between the secondary marks representing 41 and 42 is read 415, that between

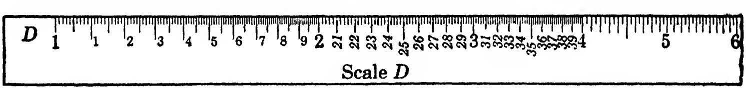


FIG. 14-3.

the secondary marks representing 55 and 56 is read 555, and the first tertiary mark to the right of the primary mark numbered 4 is read 405. The reading of any position between a pair of successive tertiary marks must be based on an estimate. Thus a position halfway between the tertiary marks associated with 222 and 224 is read 223, and a position two-fifths of the way from the tertiary mark numbered 415 to the next mark is read 417. The principle illustrated by these readings applies in all cases.

It is important to note that the decimal point has no bearing upon the position associated with a number on the *C* and *D* scales.

Consequently, the number G in Fig. 14-4 may be read 207, 2.07, 0.000207, 20,700, or any other number whose principal digits are 2, 0, and 7. The placing of the decimal point will be explained later in this chapter.

For a position between the primary marks numbered 1 and 2, four digits should be read; the first three will be exact and the last one estimated. No attempt should be made to read more than three digits for positions to the right of the primary mark numbered 4.

While making a reading, the learner should have definitely in mind the number associated with the smallest space under consideration. Thus between 1 and 2, the smallest division is associated with 10 in the fourth place; between 2 and 3, the smallest division has a value 2 in the third place; while to the right of 4, the smallest division has a value 5 in the third place.

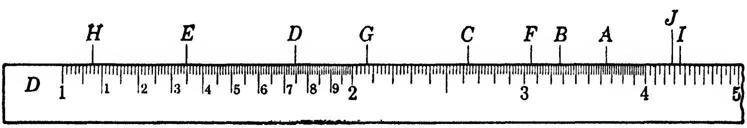


FIG. 14-4.

The learner should read from Fig. 14-4 the numbers associated with the marks lettered A, B, C, . . . and compare his readings with the following numbers: A 365, B 327, C 263, D 1745, E 1347, F 305, G 207, H 1078, I 435, J 427.

14-3. Accuracy of the slide rule. From the discussion of Art. 14-2, it appears that we read four figures of a result on one part of the scale and three figures on the remaining part. This means an attainable accuracy of roughly one part in 1000 or one-tenth of 1 per cent. The accuracy is nearly proportional to the length of the scale. Hence we associate with the 20-in. scale an accuracy of about one part in 2000, and with the Thacher cylindrical slide rule, an accuracy of about one part in 10,000. The accuracy obtainable with the 10-in. slide rule is sufficient for most practical purposes; in any case the slide rule result serves as a check.

14-4. Definitions. The central sliding part of the rule is called the **slide,** the other part, the **body.** The glass runner is called the

indicator, and the line on the indicator is referred to as the **hairline.**

The mark associated with the primary number 1 on any scale is called the **index** of the scale. An examination of the *D* scale shows that it has two indices, one at the left end and the other at the right end.

Two positions on different scales are said to be *opposite* if, without moving the slide, the hairline may be brought to cover both positions at the same time.

14-5. Multiplication. The process of multiplication may be performed by using scales *C* and *D*. The *C* scale is on the slide, but in other respects it is like the *D* scale and is read in the same manner.

To multiply **2** by 4,

> to 2 on *D* set index of *C*,
> push hairline to 4 on *C*,
> at the hairline read **8** on *D*.

Figure 14-5 shows the setting in skeleton form.

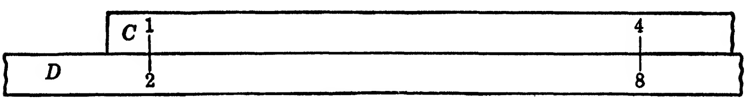


FIG. 14-5.

To multiply 3 × 3,

> to 3 on *D* set index of *C*,
> push hairline to 3 on *C*,
> at the hairline read **9** on *D*.

See Fig. 14-6 for the setting in skeleton form.

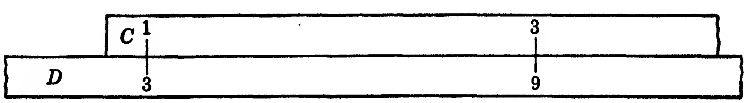


FIG. 14-6.

To multiply 1.5 × 3.5, disregard the decimal point and

> to 15 on *D* set index of *C*,
> push hairline to 35 on *C*,
> at the hairline read **525** on *D*.

By inspection we know that the answer is near 5 and is therefore **5.25.**

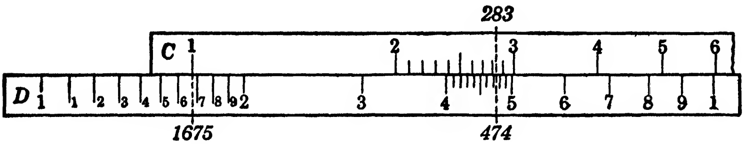


FIG. 14-7.

To find the value of 16.75 × 2.83 (see Fig. 14-7) disregard the decimal point and

> to 1675 on *D* set index of *C*,
> push hairline to 283 on *C*,
> at the hairline read **474** on *D*.

To place the decimal point we approximate the answer by noting that it is near to 3 × 16 = 48. Hence the answer is **47.4.**

These examples illustrate the use of the following rule.

Rule. To find the products of two numbers: To either number on scale *D* set index of scale *C*, push hairline to second number on scale *C* at the hairline read product on scale *D*. Disregard the decimal point while making the settings and readings; finally place the decimal point in accordance with the result of a rough approximation.

EXERCISES 14-1

1. 3 × 2.
2. 3.5 × 2.
3. 5 × 2.
4. 2 × 4.55.
5. 4.5 × 1.5.
6. 1.75 × 5.5.
7. 4.33 × 11.5.
8. 2.03 × 167.3.
9. 1.536 × 30.6.
10. 0.0756 × 1.093.
11. 1.047 × 3080.
12. 0.00205 × 408.
13. $(3.142)^2$.
14. $(1.756)^2$.

14-6. Either index may be used. It may happen that a product cannot be read when the left index of the *C* scale is used in the rule of Art. 14-5. This will be due to the fact that the second number of the product is on the part of the slide projecting beyond the body. In this case reset the slide using the right index of the *C* scale in place of the left, or use the following rule:

When a number is to be read on the *D* scale opposite a number on the slide scale and cannot be read, push the hairline to the index of the *C* scale inside the body and draw the other index of the *C* scale under the hairline. The desired reading can then be made. This very important rule applies generally.

If, to find the product of 2 and 6, we set the left index of the *C* scale opposite 2 on the *D* scale, we cannot read the answer on the *D* scale opposite 6 on the *C* scale. Hence, we set the right index of *C* opposite 2 on *D*; opposite 6 on *C* read the answer, **12,** on *D*.

Again, to find 0.0314×564,

to 314 on *D* set the right index of *C*,
push hairline to 564 on *C*,
at the hairline read **1771** on *D*.

A rough approximation is obtained by finding $0.03 \times 600 = 18$. Hence the product is **17.71.**

EXERCISES 14-2

Perform the indicated multiplications:

1. 3×5.
2. 3.05×5.17.
3. 5.56×634.
4. 743×0.0567.
5. 0.0495×0.0267.
6. 1.876×926.
7. 1.876×5.32.
8. 42.3×31.7.

14-7. Division. The process of division is performed by using the *C* and *D* scales.

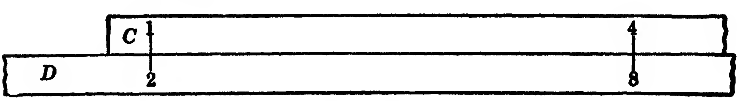


FIG. 14-8.

To divide 8 by 4 (see Fig. 14-8),

push hairline to 8 on *D*,
draw 4 of *C* under the hairline,
opposite index of *C* read **2** on *D*.

To divide 876 by 20.4

push hairline to 876 on *D*,
draw 204 of *C* under the hairline,
opposite index of *C* read **429** on *D*.

The rough calculation $800 \div 20 = 40$ shows that the decimal point must be placed after the 2. Hence the answer is **42.9.**

EXERCISES 14-3

Perform the indicated operations:

1. $87.5 \div 37.7$.
2. $3.75 \div 0.0227$.
3. $0.685 \div 8.93$.
4. $1029 \div 9.70$.
5. $0.00377 \div 5.29$.
6. $2875 \div 37.1$.
7. $871 \div 0.468$.
8. $0.0385 \div 0.001462$.
9. $3.14 \div 2.72$.
10. $3.42 \div 81.7$.

14-8. Use of scales *DF* and *CF* (folded scales). If your slide rule contains folded scales, they may often be used to save using the rule of Art. 14-6 to move the slide its own length leftward or rightward. These folded scales are used precisely like the other scales. The following rule will indicate how one may transfer operations from the *C* and *D* scales to the *CF* and *DF* scales:

Rule. Shifting an operation from the *C* and *D* scales to the *CF* and *DF* scales, or vice versa, may be made whenever the process is pushing the hairline to a number, never when a number on the slide is to be drawn under the hairline.

For example, to find 2×6,

to 2 on *D* set left index of *C*,
push hairline to 6 on *CF*,
at the hairline read **12** on *DF*.

To find 6.17×7.34,

to 617 on *DF* set index of *CF*,
push hairline to 734 on *C*,
at the hairline read **45.3** on *D*.

By using the *CF* and *DF* scales we saved the trouble of moving the slide as well as the attendant source of error. This saving, entering as it does in many ways, is a main reason for using the folded scales.

The folded scales may be used to perform multiplications and divisions just as the *C* and *D* scales are used. Thus, to find 6.17×7.34,

to 617 on *DF* set index of *CF*,
push hairline to 734 on *CF*,
at the hairline read **45.3** on *DF*;

or

to 617 on *DF* set index of *CF*,
push hairline to 734 on *C*,
at the hairline read **45.3** on *D*.

Again, to find the quotient 7.68/8.43,

push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *CF* read **0.912** on *DF*;

or

push hairline to 768 on *DF*,
draw 843 of *CF* under the hairline,
opposite the index of *C* read **0.912** on *D*.

It now appears that we may perform a multiplication or a division in several ways by using two or more of the scales *C*, *D*, *CF*, and *DF*. The rule near the beginning of this article sets forth the guiding principle. A convenient method of multiplying or dividing a number by π (= 3.14 approx.) is based on the statement: any number on *DF* is π times its opposite on *D*, and any number on *D* is $1/\pi$ times its opposite on *DF*.

EXERCISES 14-4

Perform each of the operations indicated in Exercises 1 to 11 in four ways: (1) by using the *C* and *D* scales only; (2) by using the *CF* and *DF* scales only; (3) by using the *C* and *D* scales for the initial setting and the *CF* and *DF* scales for completing the solution; (4) by using the *CF* and *DF* scales for the initial setting and the *C* and *D* scales for completing the solution.

1. 5.78×6.35.
2. 7.84×1.065.
3. $0.00465 \div 73.6$.
4. $0.0634 \times 53{,}600$.
5. $1.769 \div 496$.
6. $946 \div 0.0677$.
7. 813×1.951.
8. $0.00755 \div 0.338$.
9. $0.0948 \div 7.23$.
10. $149.0 \div 63.3$.
11. $2.718 \div 65.7$.
12. 783π.
13. $783 \div \pi$.
14. 0.0876π.
15. $0.504 \div \pi$.
16. $1.072 \div 10.97$.

14-9. The proportion principle. The proportion principle is very important because settings can be devised and recalled by using it. **When the slide is set in any position, the ratio of any number on the D scale to its opposite on the C scale is the same as the ratio of any other number on D to its opposite on C.** For example, draw 1 of C opposite 2 on D and find the opposites indicated in the following table:

C (or CF)	1	1.5	2.5	3	4	5
D (or DF)	2	3	5	6	8	10

Now consider the proportion

$$\frac{x}{56} = \frac{9}{7}. \tag{1}$$

If 9 on C be set opposite 7 on D, then x will appear on C opposite 56 on D. Hence, to find x in (1),

push hairline to 7 on D,
draw 9 of C under the hairline,
push hairline to 56 on D,
at the hairline read **72** on C,

or

push hairline to 9 on D,
draw 7 of C under the hairline,
push hairline to 56 on C,
at the hairline read **72** on D.

Again, consider the continued proportion

$$\frac{C}{D}: \quad \frac{3.15}{5.29} = \frac{x}{4.35} = \frac{57.6}{y} = \frac{z}{183.4}.$$

Observe that 3.15/5.29 is the known ratio, and

push hairline to 529 on D,
draw 315 of C under the hairline;
opposite 435 on D, read x = **2.59** on C,
opposite 576 on C, read y = **96.7** on D,
opposite 1834 on D, read z = **109.2** on C.

The positions of the decimal points were determined by noticing that each denominator had to be approximately twice its numerator since 5.29 is approximately twice 3.15. The position of the decimal point is always determined by a rough approximation.

Whenever an answer cannot be read because the slide projects beyond the body, use the rules of Arts. 14-6 and 14-8.

EXERCISES 14-5

Find, in each of the following equations, the values of the unknowns:

1. $\dfrac{2}{3} = \dfrac{x}{7.83}$.

2. $\dfrac{x}{1.804} = \dfrac{y}{25} = \dfrac{1}{0.785}$.

3. $\dfrac{x}{709} = \dfrac{246}{y} = \dfrac{28}{384}$.

4. $\dfrac{x}{0.204} = \dfrac{y}{0.506} = \dfrac{5.28}{z} = \dfrac{2.01}{0.1034}$.

5. $\dfrac{x}{2.07} = \dfrac{3}{61.3} = \dfrac{z}{1.571}$.

6. $\dfrac{8.51}{1.5} = \dfrac{9}{x} = \dfrac{235}{y}$.

7. $\dfrac{17}{x} = \dfrac{1.365}{8.53} = \dfrac{4.86}{y}$.

8. $\dfrac{x}{y} = \dfrac{y}{7.34} = \dfrac{3.75}{29.7}$.

9. $\dfrac{x}{49.6} = \dfrac{z}{y} = \dfrac{y}{3.58} = \dfrac{1.076}{0.287}$.

14-10. Use of the *CI* scale. The scale marked *CI* is designed so that **when the hairline is set to a number on the *CI* scale, its reciprocal** (1 divided by the number) **is set on the *C* scale.** Accordingly this scale may be used to deal with reciprocals. Thus, to find x when

$$x = 415 \times 1.87 \times 2.54,$$

divide through by 415 and replace 2.54 by $1 \div (1/2.54)$ to get

$$\frac{D}{C}: \qquad \frac{x}{415} = \frac{1.87}{1/2.54}.$$

Hence, in accordance with the proportion principle,

push hairline to 1.87 on *D*,
draw 2.54 of *CI* under the hairline,
push hairline to 415 on *C*,
at the hairline read $x =$ **1970** on *D*.

Observe that 1/2.54 of C was drawn under the hairline indirectly by drawing 2.54 on CI under the hairline. If one keeps in mind the statement in boldface, he will find that he can multiply by the reciprocal of a number, divide by it, or use it in a proportion by using the CI scale for the number instead of the C scale. The same principle governs the use of the CIF scale.

EXERCISES 14-6

In each of the following equations find the value of the unknown:

1. $\dfrac{y}{28} = \dfrac{3.2}{\frac{1}{118}}$.
2. $\dfrac{y}{42} = \dfrac{39.2}{\frac{1}{56}}$.
3. $y = 25(\frac{1}{712})$.
4. $y = 74.5\left(\dfrac{1}{42.3}\right)$.
5. $y = (321)(46.2)(4.93)$.
6. $y = (62)(49)(82)$.
7. $(36.2)(47.2)y = 3.8$.
8. $y = \dfrac{3.41}{(1.72)(6.31)}$.
9. $y = \dfrac{(6.72)}{(5.81)(6.43)}$.
10. $y = (\frac{1}{6})(14)(\frac{1}{15})$.

14-11. Combined multiplication and division. The importance of this article is secondary only to Art. 14-9, which relates to the proportion principle.

Example 1. Find the value of $\dfrac{7.36 \times 8.44}{92}$.

Solution. Reason as follows: first divide 7.36 by 92, and then multiply the result by 8.44. This would suggest that we

push hairline to 736 on D,
draw 92 of C under the hairline;
opposite 8.44 on C, read **0.675** on D.

Example 2. Find the value of $\dfrac{18 \times 45 \times 37}{23 \times 29}$.

Solution. Reason as follows: (*a*) divide by 18 by 23, (*b*) multiply the result by 45, (*c*) divide this second result by 29, (*d*) multiply this third result by 37. This argument suggests that we

push hairline to 18 on D,
draw 23 of C under the hairline,
push hairline to 45 on C,
draw 29 of C under the hairline,
push hairline to 37 on C,
at the hairline read **449** on D.

To determine the position of the decimal point write $\frac{20 \times 40 \times 40}{20 \times 30}$ = about 50. Hence the answer is **44.9.**

A little reflection on the procedure of Example 2 will enable the operator to evaluate by the shortest method expressions similar to the one just considered. He should observe that: the D scale was used only twice, once at the beginning of the process and once at its end; *the process for each number of the denominator consisted in drawing that number, located on the C scale, under the hairline; the process for each number of the numerator consisted in pushing the hairline to that number located on the C scale.*

If at any time the indicator cannot be placed because of the projection of the slide, apply the rule of Art. 14-6, *or carry on the operations using the folded scales.*

Example 3. Find the value of $1.843 \times 92 \times 2.45 \times 0.584 \times 365$.

Solution. Write the given expression in the form

$$\frac{1.843 \times 2.45 \times 365}{(1/92)(1/0.584)}$$

and reason as follows: (a) divide 1.843 by (1/92), (b) multiply the result by 2.45, (c) divide this second result by (1/0.584), (d) multiply the third result by 365. This argument suggests that we

push hairline to 1843 on D,
draw 92 of CI under the hairline,
push hairline to 245 on C,
draw 584 of CI under the hairline,
push hairline to 365 on C,
at the hairline read **886** on D.

To approximate the answer we write

$$2(90)\left(\frac{5}{2}\right)\left(\frac{6}{10}\right)300 = 81{,}000.$$

Hence the answer is **88,600.**

EXERCISES 14-7

1. $\dfrac{1375 \times 0.0642}{76{,}400}$.
2. $\dfrac{45.2 \times 11.24}{336}$.
3. $\dfrac{218}{4.23 \times 50.8}$.
4. $\dfrac{235}{3.86 \times 3.54}$.
5. $2.84 \times 6.52 \times 5.19$.
6. $9.21 \times 0.1795 \times 0.0672$.
7. $37.7 \times 4.82 \times 830$.
8. $\dfrac{65.7 \times 0.835}{3.58}$.
9. $\dfrac{362}{3.86 \times 9.61}$.
10. $\dfrac{24.1}{261 \times 32.1}$.
11. $\dfrac{75.5 \times 63.4 \times 95}{3.14}$.
12. $\dfrac{3.97}{51.2 \times 0.925 \times 3.14}$.
13. $\dfrac{47.3 \times 3.14}{32.5 \times 16.4}$.
14. $\dfrac{3.82 \times 6.95 \times 7.85 \times 436}{79.8 \times 0.0317 \times 870}$.
15. $187 \times 0.00236 \times 0.0768 \times 1047 \times 3.14$.
16. $\dfrac{0.917 \times 8.65 \times 1076 \times 3152}{7840}$.

14-12. Square roots. The square root of a given number is a second number whose square is the given number. Thus the square root of 4 is 2, and the square root of 9 is 3, or, using the symbol for square root, $\sqrt{4} = 2$, and $\sqrt{9} = 3$.

The *A* scale consists of two parts that differ only in slight details. We shall refer to the left-hand part as *A left* and to the right-hand part as *A right*. Similar reference will be made to the *B* scale.

Rule. To find the square root of a number between 1 and 10, set the hairline to the number on scale *A* left and read its square root at the hairline on the *D* scale. To find the square root of a number between 10 and 100, set the hairline to the number on scale *A* right and read its square root at the hairline on the

D scale. In either case place the decimal point after the first digit. A similar statement relating to the *B* scale and the *C* scale holds true. For example, set the hairline to 9 on scale *A* left, read 3 ($= \sqrt{9}$) at the hairline on *D*, set the hairline to 25 on scale *B* right, read 5 ($= \sqrt{25}$) at the hairline on *C*.

To obtain the square root of any number, *move the decimal point an even number of places to obtain a number between* 1 *and* 100; *then apply the rule above; finally move the decimal point one half as many places as it was moved in the original number but in the opposite direction.** The learner may also place the decimal point in accordance with information derived from a rough approximation.

For example, to find the square root of 23,400, move the decimal point four places to the left, thus getting 2.34 (a number between 1 and 10); set the hairline to 2.34 on scale *A* left; read 1.530 at the hairline on the *D* scale; finally, move the decimal point one-half of 4 or two places to the right to obtain the answer **153.0.** The decimal point could have been placed after observing that $\sqrt{10{,}000} = 100$ or that $\sqrt{40{,}000} = 200$. Also, the left *B* scale and the *C* scale could have been used instead of the left *A* scale and the *D* scale.

To find $\sqrt{3850}$, move the decimal point two places to the left to obtain $\sqrt{38.50}$; set the hairline to 38.50 on scale *A* right; read 6.20 at the hairline on the *D* scale; move the decimal point one place to the right to obtain the answer **62.0.** The decimal point could have been placed by observing that $\sqrt{3600} = 60$.

To find $\sqrt{0.000585}$, move the decimal point four places to the right to obtain $\sqrt{5.85}$; find $\sqrt{5.85} = 2.42$; move the decimal point two places to the left to obtain the answer **0.0242.**

EXERCISES 14-8

1. Find the square root of each of the following numbers: 8, 12, 17, 89, 8.90, 890, 0.89, 7280, 0.0635, 0.0000635, 63,500, 100,000.

2. Find the length of the side of a square whose area is (*a*) 53,500 ft.^{2}; (*b*) 0.0776 ft.^{2}; (*c*) 3.27×10^7 ft.^{2}

* The following rule may also be used: If the square root of a number greater than unity is desired, use *A* left when it contains an odd number of digits to the left of the decimal point; otherwise use *A* right. For a number less than unity use *A* left if the number of zeros immediately following the decimal point is odd; otherwise, use *A* right.

3. Find the diameter of a circle having area (*a*) 256 ft.^{2}; (*b*) 0.773 ft.^{2}; (*c*) 1950 ft.^{2}

14-13. Combined operations involving square roots. When the hairline is set to a number on the *B* scale, it is automatically set on the *C* scale to the square root of the number. Therefore the *B* scale can be used in combined operations like the *CI* scale. Naturally, the rule for square-root settings should be used to determine whether *B* left or *B* right is to be used in any particular case. The following example will illustrate the method of procedure.

Example. Evaluate $\dfrac{\sqrt{832} \times \sqrt{365} \times 1863}{(\frac{1}{736}) \times 89{,}400}$.

Solution. In accordance with italicized statement of Art. 14-11,

push hairline to 832 on *A* left,
draw 736 of *CI* under the hairline,
push hairline to 365 on *B* left,
draw 894 of *C* under the hairline,
push hairline to 1863 on *CF*,
at the hairline read **8450** on *DF*.

The method of finding cube roots is much like that of finding square roots. The following rule may be used:

Rule. To obtain the cube root of a number, move the decimal point over three places (or digits) at a time until a number between 1 and 1000 is obtained. Then push the hairline to the new number on *K* left, *K* middle, or *K* right according as it lies between 1 and 10, 10 and 100, or 100 and 1000. Read the cube root on scale *D* at the hairline and place the decimal point after the first digit. Then move the decimal point one-third as many places as it was moved in the original number but in the opposite direction.

EXERCISES 14-9

1. $\dfrac{7.87 \times \sqrt{377}}{2.38}$.

2. $\dfrac{4.25 \times \sqrt{63.5} \times \sqrt{7.75}}{0.275 \times \pi}$.

3. $\dfrac{86 \times \sqrt{734} \times \pi}{775 \times \sqrt{0.685}}$.

4. $\dfrac{(2.60)^2}{2.17 \times 7.28}$.

5. $\dfrac{20.6 \times 7.89^2 \times 6.79^2}{4.67^2 \times 281}$.

6. $\dfrac{189.7 \times \sqrt{0.00296} \times \sqrt{347} \times 0.274}{\sqrt{2.85} \times 165 \times \pi}$.

7. $\sqrt{285} \times 667 \times \sqrt{6.65} \times 78.4 \times \sqrt{0.00449}$.

8. $\dfrac{239 \times \sqrt{0.677} \times 374 \times 9.45 \times \pi}{84.3 \times \sqrt{9350} \times \sqrt{28400}}$.

14-14. The *S* (sine) and *ST* (sine tangent) scales. The numbers on the sine scales *S* and *ST** represent angles. In order to set the indicator to an angle on the sine scales it is necessary to

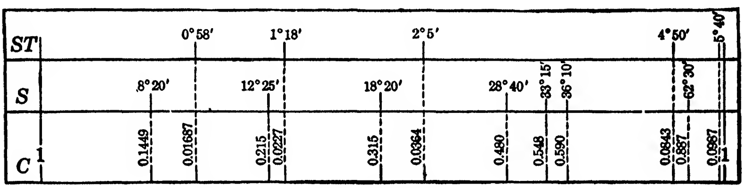


FIG. 14-9.

determine the value of the angles represented by the subdivisions. Thus, since there are six primary intervals between 4° and 5°, each represents 10′; since each of the primary intervals is subdivided into five secondary intervals, each of the latter represents 2′. Again, since there are five primary intervals between 20° and 25°, each represents 1°; since each primary interval here is subdivided into two secondary intervals, each of the latter represents 30′; since each secondary interval is subdivided into three parts, these smallest intervals represent 10′. These illustrations indicate the manner in which the learner should analyze the part of the scale involved to find the value of the smallest interval to be considered. In general, when setting the hairline to an angle, the student should always have in mind the value of the smallest interval on the part of the slide rule under consideration.

When the indicator is set to a black number (angle) on scale *S* or *ST*, the sine of the angle is on scale *C* at the hairline and hence on scale *D* when the indices on scales *C* and *D* coincide.

When scale *C* is used to read sines of angles on *ST*, the left index of *C* is taken as 0.01, the right index as 0.1. In reading sines

* The *ST* scale is a sine scale, but since it is also used as a tangent scale it is designated *ST*.

of angles on S, the left index of C is taken as 0.1, the right index as 1. Thus, to find sin 36°26′, opposite 36°26′ on scale S, read 0.594 on scale C; to find sin 3°24′, opposite 3°24′ on scale ST, read 0.0593 on scale C. Figure 14-9 shows scales ST, S, and C on which certain angles and their sines are indicated. As an exercise, read from your slide rule the sines of the angles shown in the figure and compare your results with those given.

EXERCISES 14-10

1. By examination of the slide rule verify that on the S scale from the left index to 16° the smallest subdivision represents 5′; from 16° to 30° it represents 10′; from 30° to 60° it represents 30′; from 60° to 80° it represents 1°; and from 80° to 90° it represents 5°.

2. Find the sine of each of the following angles:

(*a*) 30°. (*b*) 38°. (*c*) 3°20′. (*d*) 90°. (*e*) 87°45′.
(*f*) 1°35′. (*g*) 14°38′. (*h*) 22°25′. (*i*) 11°48′. (*j*) 51°30′.

3. Find the cosine of each of the angles in Exercise 2 by using the relation $\cos \varphi = \sin (90° - \varphi)$.

4. For each of the following values of x,

(*a*) 0.5, (*b*) 0.875, (*c*) 0.375, (*d*) 0.1, (*e*) 0.015,
(*f*) 0.62, (*g*) 0.062, (*h*) 0.031, (*i*) 0.92, (*j*) 0.885,

find the value of φ less than 90°, (*A*) if $\varphi = \sin^{-1} x$, where $\sin^{-1} x$ means "the angle whose sine is x"; (*B*) if $\varphi = \cos^{-1} x$.

5. Find the cosecant of each of the angles in Exercise 2 by using the relation $\csc \varphi = \dfrac{1}{\sin \varphi}$.

Hint. Set the angle on S, read the cosecant on CI (or on DI when the rule is closed).

6. Find the secant of each of the angles in Exercise 2 by using the relation $\sec \varphi = \dfrac{1}{\cos \varphi}$.

7. For each of the following values of x,

(*a*) 2, (*b*) 2.4, (*c*) 1.7, (*d*) 6.12, (*e*) 80.2, (*f*) 4.72,

find the value of φ less than 90°, (*A*) if $\varphi = \csc^{-1} x$; (*B*) if $\varphi = \sec^{-1} x$.

14-15. The *T* (tangent) scale. When the indicator is set to a black angle on scale *T*, the tangent of the angle is on scale *C* at

the hairline and hence on scale D when the indices of scales T and D coincide. Also **when the indicator is set to a black angle on scale *T*, the cotangent of the angle is on scale *CI* at the hairline.** Thus, to find tan 36°, push the hairline to 36° on T; at the hairline read **0.727** on C. To find cot 27°10′, push the hairline to 27°10′ on T; at the hairline read **1.949** on CI.

When scale C is used to read tangents, the left index is taken as 0.1 and the right index as 1.0. Only those angles that range from 5°43′ to 45° appear on scale T. It is shown in trigonometry that for angles less than 5°43′, the sine and tangent are approximately equal. Hence, so far as the slide rule is concerned, the tangent of an angle less than 5°43′ may be replaced by the sine of the angle. Thus to find tan 2°15′, push the hairline to 2°15′ on ST, at the hairline read **0.0393** on C. To find the tangent of an angle greater than 45°, use the relation

$$\cot \theta = \tan (90° - \theta).$$

To find tan 56°, push the hairline to 34° (= 90° − 56°) on T, at the hairline read **1.483** on CI. The student should observe that he could have set the hairline to 56° in red on the T scale and thus have avoided subtracting 34° from 90°.

EXERCISES 14-11

1. Fill out the following table:

φ	8°6′	27°15′	62°19′	1°7′	74°15′	87°	47°28′
tan φ							
cot φ							

2. The following numbers are tangents of angles. Find the angles.

(*a*) 0.24. (*b*) 0.785. (*c*) 0.92. (*d*) 0.54. (*e*) 0.059.
(*f*) 0.082. (*g*) 0.432. (*h*) 0.043. (*i*) 0.0149. (*j*) 0.374.
(*k*) 3.72. (*l*) 4.67. (*m*) 17.01. (*n*) 1.03. (*o*) 1.232.

3. The numbers in Exercise 2 are cotangents of angles. Find the angles.

14-16. Combined operations. The method for evaluating expressions involving combined operations as stated in Arts. 14-11

and 14-13 applies without change when some of the numbers are trigonometric functions. This is illustrated in the following example.

Example. Evaluate $\dfrac{6.1\sqrt{17}\sin 72^\circ \tan 20^\circ}{2.2}$.

Solution. Write

$$\frac{\sqrt{17}\sin 72^\circ \tan 20^\circ}{2.2\left(\dfrac{1}{6.1}\right)}.$$

Push hairline to 17 on A right,
draw 2.2 of C under the hairline,
push hairline to 20° on T,
draw 6.1 of CI under the hairline,
push hairline to 72° on S,
at the hairline read **3.96** on D.

EXERCISES 14-12

Evaluate the following:

1. $\dfrac{18.6\sin 36^\circ}{\sin 21^\circ}$.

2. $\dfrac{32\sin 18^\circ}{27.5}$.

3. $\dfrac{4.2\tan 38^\circ}{\sin 45^\circ 30'}$.

4. $\dfrac{34.3\sin 17^\circ}{\tan 22^\circ 30'}$.

5. $\dfrac{13.1\cos 40^\circ}{\tan 35^\circ 10'}$.

6. $\dfrac{17.2\cos 35^\circ}{\cot 50^\circ}$.

7. $\dfrac{7.8\csc 35^\circ 30'}{\cot 21^\circ 25'}$.

8. $\dfrac{63.1\sec 80^\circ}{\tan 55^\circ}$.

9. $\dfrac{\sin 18^\circ \tan 20^\circ}{3.7\tan 41^\circ \sin 31^\circ}$.

10. $\dfrac{\sin 26^\circ 25'}{8.1\tan 22^\circ 18'}$.

11. $3.14\sin 13^\circ 10' \csc 32^\circ$.

12. $7.1\pi \sin 47^\circ 35'$.

13. $\dfrac{0.61\csc 12^\circ 15'}{\cot 35^\circ 16'}$.

14. $\dfrac{1\sin 22^\circ 40'}{\tan 28^\circ 10'}$.

15. $\dfrac{3.1\sin 61^\circ 35' \csc 15^\circ 18'}{\cos 27^\circ 40' \cot 20^\circ}$.

16. $\dfrac{13.1\sin 3^\circ 7'}{\tan 30^\circ 10'}$.

17. $\dfrac{0.0037\sin 49^\circ 50'}{\tan 2^\circ 6'}$.

18. $\dfrac{\sqrt{16.5}\sin 45^\circ 30'}{\sqrt{4.6}\,41.2\cot 71^\circ 10'}$.

19. $\dfrac{\sqrt[3]{6.1}\,4.91}{\tan 13^\circ 14'}$.

20. $\dfrac{\sin 51^\circ 30'}{(39.1)(6.28)}$.

21. $\dfrac{\csc 49°30'}{(19.1)(7.61)\sqrt{69.4}}$.

22. $(48.1)(1.68) \sin 39°$.

23. $0.0121 \sin 81° \cot 41°$.

24. $\dfrac{1.01 \cos 71°10' \sin 15°}{\sqrt{4.81} \cos 27°10'}$.

14-17. Solving a triangle by means of the law of sines. If the sides and angles of a triangle are lettered as indicated in Fig. 14-10, the law of sines is written

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{2}$$

This law is the basis of most slide-rule solutions of triangles.

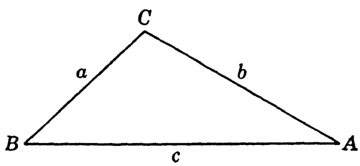


FIG. 14-10.

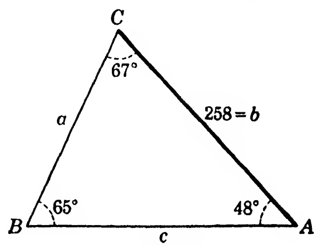


FIG. 14-11.

To solve the triangle shown in Fig. 14-11 for a and c, write

$$\frac{\sin 65°}{258} = \frac{\sin 48°}{a} = \frac{\sin 67°}{c},$$

and, using the setting based on the proportion principle,

push hairline to 258 on D,
draw 65° of S under the hairline,
push hairline to 48° on S,
at the hairline read a = **212** on D,
push hairline to 67° on S,
at the hairline read c = **262** on D.

The decimal point was placed by inspection.

In general, to solve any triangle in which a side and the angle opposite are known,

push hairline to known side on *D*,
draw opposite angle of *S* under the hairline,
push hairline to any known side on *D*,

at the hairline read opposite angle on S,
push hairline to any known angle on S,
at the hairline read opposite side on D.

When an angle A of a triangle is greater than 90°, replace it by $180° - A$. This is permissible since $\sin(180° - A) = \sin A$. When the decimal point in a result cannot be placed by inspection, compute the part involved approximately by using (2) with the trigonometric functions replaced by their natural values.

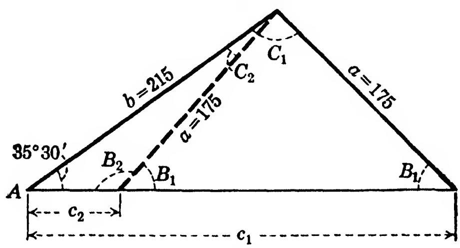


FIG. 14-12.

When the given parts of a triangle are two sides and the angle opposite one of them, there may be two solutions. For example, if the given parts are $a = 175$, $b = 215$, $A = 35°30'$, the two possible triangles are shown in Fig. 14-12. Using the setting (2) of Art. 14-17,

push hairline to 175 on D,
draw 35°30′ of S under the hairline,
push hairline to 215 on D,
at the hairline read $B_1 = \mathbf{45°30'}$ on S.
Compute $C_1 = 180° - 35°30' - 45°30' = 99°$
push hairline to 81° (= 180° − 99°) on S,
at the hairline read $c_1 = \mathbf{298}$ on D.
Compute $C_2 = B_1 - 35°30' = \mathbf{10°}$,
push hairline to 10° on S,
at the hairline read $c_2 = \mathbf{52.3}$ on D.

EXERCISES 14-13

Solve the following oblique triangles:

1. $a = 50$,	**2.** $c = 60$,	**3.** $a = 550$,
$A = 65°$,	$A = 50°$,	$A = 10°12'$,
$B = 40°$.	$B = 75°$.	$B = 46°36'$.

4. $a = 222$, $b = 4570$, $C = 90°$.

5. $a = 120$, $b = 80$, $A = 60°$.

6. $b = 0.234$, $c = 0.198$, $B = 109°$.

7. $a = 795$, $A = 79°59'$, $B = 44°41'$.

8. $a = 21$, $A = 4°10'$, $B = 75°$.

9. $b = 91.1$, $c = 77$, $B = 51°7'$.

10. $a = 50$, $c = 66$, $A = 123°11'$.

11. $a = 8.66$, $c = 10$, $A = 59°57'$.

12. $b = 8$, $a = 120$, $A = 60°$.

13. A ship at point S can be seen from each of two points, A and B, on the shore. If $AB = 800$ ft., angle $SAB = 67°43'$, and angle

$$SBA = 74°21',$$

find the distance of the ship from A.

14. To determine the distance of an inaccessible tower A from a point B, a line BC and the angles ABC and BCA were measured and found to be 1000 yd., 44°, and 70°, respectively. Find the distance AB.

Solve the following oblique triangles:

15. $a = 18$, $b = 20$, $A = 55°24'$.

16. $b = 19$, $c = 18$, $C = 15°49'$.

17. $a = 32.2$, $c = 27.1$, $C = 52°24'$.

18. $b = 5.16$, $c = 6.84$, $B = 44°3'$.

19. $a = 177$, $b = 216$, $A = 35°36'$.

20. $a = 17,060$, $b = 14,050$, $B = 40°$.

21. Find the length of the perpendicular p for the triangle of Fig. 14-13. How many solutions will there be for triangle ABC if (*a*) $b = 3$? (*b*) $b = 4$? (*c*) $b = p$?

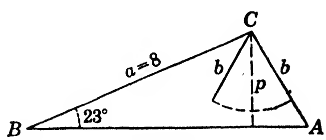


FIG. 14-13.

14-18. To solve a right triangle when two legs are given. When the two legs of a right triangle are the given parts, first find the smaller acute angle from its tangent, and then apply the law of sines to complete the solution.

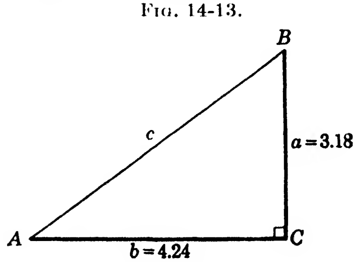


FIG. 14-14.

Example. Solve the right triangle of Fig. 14-14 in which $a = 3.18$, $b = 4.24$.

Solution. From the triangle read $\tan A = \frac{3.18}{4.24}$, and write this equation in the form

$$\frac{\tan A}{3.18} = \frac{1}{4.24}.$$

Using the setting based on the principle of proportion,

set index of C to 4.24 on D,
push hairline to 3.18 on D,
at the hairline read A = **36°52′** on T.

Since angle $A = 36°52'$ and $a = 3.18$, we know a pair of opposite parts and may proceed to use the law of sines. Since the hairline is on 3.18 of D from the setting just made,

draw 36°52′ of S under the hairline,
at index of C read c = **5.31** on D.
Evidently $B = 90° - A$ = **53°8′**.

The following rule states the method of solution:

Rule. To solve a right triangle for which two legs are given,

set index of C to larger leg on D,
push hairline to smaller leg on D,
at the hairline read the smaller acute angle on T,
draw this angle on S under the hairline,
at index of slide read hypotenuse on D.

EXERCISES 14-14

Solve the following right triangles:

1. $a = 12.3$, $b = 20.2$.	**2.** $a = 273$, $b = 418$.	**3.** $a = 13.2$, $b = 13.2$.
4. $a = 101$, $b = 116$.	**5.** $a = 28$, $b = 34$.	**6.** $a = 42$, $b = 71$.
7. $a = 50$, $b = 23.3$.	**8.** $a = 12$, $b = 5$.	**9.** $a = 0.31$, $b = 4.8$.

14-19. To solve a triangle in which two sides and the included angle are given. The method here explained will consist in dividing the given triangle into two right triangles by means of

an altitude to one of the known sides and then solving the two right triangles separately. The method is illustrated in the following example.

Example. Solve the triangle of Fig. 14-15 in which $a = 6.18$, $b = 9.27$, $C = 32°$.

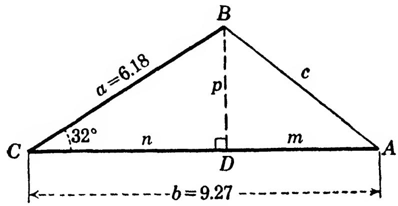


FIG. 14-15.

Solution. Draw the altitude BD to side AC, and observe that angle $BCD = 90°$ and $a = 6.18$ are known. Hence use the rule of Art. 14-17 and

set index of C to 6.18 on D,
push hairline to 32° on S,
at the hairline read $p = 3.27$ on D,
opposite 58° ($= 90° - 32°$) on S read $n = 5.24$ on D,
compute $m = 9.27 - 5.24 = 4.03$.

To solve triangle ABD, use the rule of Art. 14-18. Hence

set index of C to 4.03 on D,
push hairline to 3.27 on D,
at the hairline read $A =$ **39°3′** on T,
draw 39°3′ on S under the hairline,
at index of C read $c =$ **5.19** on D.
Evidently $B = 180° - 32° - 39°3' =$ **108°57′**.

If the given angle is obtuse, the altitude lies outside the triangle, but the method is essentially the same as that used in the solution above.

EXERCISES 14-15

Solve the following triangles:

1. $a = 94$,	**2.** $a = 100$,	**3.** $a = 235$,
$b = 56$,	$c = 130$,	$b = 185$,
$C = 29°$.	$B = 51°49'$.	$C = 84°36'$.

4. $b = 2.30$, $c = 3.57$, $A = 62°$.

5. $a = 27$, $c = 15$, $B = 46°$.

6. $a = 6.75$, $c = 1.04$, $B = 127°9'$.

7. $a = 0.085$, $c = 0.0042$, $B = 56°30'$.

8. $a = 17$, $b = 12$, $C = 59°18'$.

9. $b = 2580$. $c = 5290$, $A = 138°21'$.

10. The two diagonals of a parallelogram are 10 and 12 and they form an angle of 49°18′. Find the length of each side.

11. Two ships start from the same point at the same instant. One sails due north at the rate of 10.44 miles per hour, and the other due northeast at the rate of 7.71 miles per hour. How far apart are they at the end of 40 min.?

14-20. To solve a triangle in which three sides are given. When three sides of a triangle are given, one angle may be found by using the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

and the other parts may then be found by means of the law of sines.

Example. Solve the triangle of Fig. 14-16 in which $a = 15$, $b = 18$, $c = 20$.

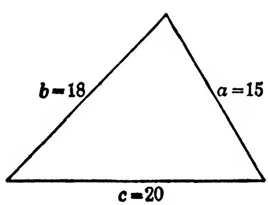


FIG. 14-16.

Solution. From the law of cosines we write

$$\frac{\cos A}{1} = \frac{b^2 + c^2 - a^2}{2bc} = \frac{18^2 + 20^2 - 15^2}{2 \times 18 \times 20} = \frac{499}{720}.$$

Hence, using a setting based on the proportion principle,

to 720 on D set 499 of C,
at index of D read A = **46°6′** on S (red).

Now complete the solution by means of the law of sines to obtain B = **59°54′**, C = **74°**. When all three angles are read from the slide rule, the relation $A + B + C = 180°$ may be used as a check. Thus, for the solution just completed,

$$A + B + C = 46°6' + 59°54' + 74° = 180°.$$

EXERCISES 14-16

Solve the following triangles:

1. $a = 3.41$, $b = 2.60$, $c = 1.58$.	**2.** $a = 35$, $b = 38$, $c = 41$.	**3.** $a = 97.9$, $b = 106$, $c = 139$.
4. $a = 111$, $b = 145$, $c = 40$.	**5.** $a = 61.0$, $b = 49.2$, $c = 80.5$.	**6.** $a = 57.9$, $b = 50.1$, $c = 35.0$.

14-21. To change radians to degrees or degrees to radians. Since π (= 3.1416 approx.) radians equal 180°, we may write

$$\frac{\pi}{180} = \frac{r \text{ (number of radians)}}{d \text{ (number of degrees)}}.$$

Hence

draw π on C opposite 180 on D,
push hairline to d (number of degrees given) on D,
at the hairline read number of radians on C,
push hairline to r (number of radians given) on C,
at the hairline read number of degrees on D.

EXERCISES 14-17

1. Express the following angles in radians:

(*a*) 45°.	(*b*) 60°.	(*c*) 90°.
(*d*) 180°.	(*e*) 120°.	(*f*) 135°.
(*g*) 22°30′.	(*h*) 200°.	(*i*) 3000°.

2. Express the following angles in degrees:

(*a*) $\pi/3$ radians.	(*b*) $3\pi/4$ radians.	(*c*) $\pi/72$ radian.
(*d*) $7\pi/6$ radians.	(*e*) $20\pi/3$ radians.	(*f*) 0.98π radians.

3. Express in radians the following angles:

(*a*) 1°.	(*b*) 1′.	(*c*) 1″.
(*d*) 10°11′.	(*e*) 180°34′20″.	(*f*) 300°25′43″.

4. Find the following angles in degrees and minutes:

(*a*) $\frac{1}{10}$ radian; (*b*) $2\frac{1}{2}$ radians; (*c*) 1.6 radians; (*d*) 6 radians.

TABLES

Table I.—Common Logarithms

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

TABLE I.—COMMON LOGARITHMS—*Continued*

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N.	0	1	2	3	4	5	6	7	8	9

TABLE II.—TRIGONOMETRIC FUNCTIONS

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
0° 00′	.0000	∞	1.0000	0.0000	.0000	∞	0	∞	90° 00′
10	.0029	7.4637	1.0000	0000	.0029	7.4637	343.77	2.5363	50
20	.0058	7648	1.0000	0000	.0058	7648	171.89	2352	40
30	.0087	9408	1.0000	0000	.0087	9409	114.59	0591	30
40	.0116	8.0658	.9999	0000	.0116	8.0658	85.940	1.9342	20
50	.0145	1627	.9999	0000	.0145	1627	68.750	8373	10
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	89° 00′
10	.0204	3088	.9998	9999	.0204	3089	49.104	6911	50
20	.0233	3668	.9997	9999	.0233	3669	42.964	6331	40
30	.0262	4179	.9997	9999	.0262	4181	38.188	5819	30
40	.0291	4637	.9996	9998	.0291	4638	34.368	5362	20
50	.0320	5050	.9995	9998	.0320	5053	31.242	4947	10
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	88° 00′
10	.0378	5776	.9993	9997	.0378	5779	26.432	4221	50
20	.0407	6097	.9992	9996	.0407	6101	24.542	3899	40
30	.0436	6397	.9990	9996	.0437	6401	22.904	3599	30
40	.0465	6677	.9989	9995	.0466	6682	21.470	3318	20
50	.0494	6940	.9988	9995	.0495	6945	20.206	3055	10
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	87° 00′
10	.0552	7423	.9985	9993	.0553	7429	18.075	2571	50
20	.0581	7645	.9983	9993	.0582	7652	17.169	2348	40
30	.0610	7857	.9981	9992	.0612	7865	16.350	2135	30
40	.0640	8059	.9980	9991	.0641	8067	15.605	1933	20
50	.0669	8251	.9978	9990	.0670	8261	14.924	1739	10
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	86° 00′
10	.0727	8613	.9974	9989	.0729	8624	13.727	1376	50
20	.0756	8783	.9971	9988	.0758	8795	13.197	1205	40
30	.0785	8946	.9969	9987	.0787	8960	12.706	1040	30
40	.0814	9104	.9967	9986	.0816	9118	12.251	0882	20
50	.0843	9256	.9964	9985	.0846	9272	11.826	0728	10
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	85° 00′
10	.0901	9545	.9959	9982	.0904	9563	11.059	0437	50
20	.0929	9682	.9957	9981	.0934	9701	10.712	0299	40
30	.0958	9816	.9954	9980	.0963	9836	10.385	0164	30
40	.0987	9945	.9951	9979	.0992	9966	10.078	0034	20
50	.1016	9.0070	.9948	9977	.1022	9.0093	9.7882	0.9907	10
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	84° 00′
10	.1074	0311	.9942	9975	.1080	0336	9.2553	9664	50
20	.1103	0426	.9939	9973	.1110	0453	9.0098	9547	40
30	.1132	0539	.9936	9972	.1139	0567	8.7769	9433	30
40	.1161	0648	.9932	9971	.1169	0678	8.5555	9322	20
50	.1190	0755	.9929	9969	.1198	0786	8.3450	9214	10
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	83° 00′
10	.1248	0961	.9922	9966	.1257	0995	7.9530	9005	50
20	.1276	1060	.9918	9964	.1287	1096	7.7704	8904	40
30	.1305	1157	.9914	9963	.1317	1194	7.5958	8806	30
40	.1334	1252	.9911	9961	.1346	1291	7.4287	8709	20
50	.1363	1345	.9907	9959	.1376	1385	7.2687	8615	10
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	82° 00′
10	.1421	1525	.9899	9956	.1435	1569	6.9682	8431	50
20	.1449	1612	.9894	9954	.1465	1658	6.8269	8342	40
30	.1478	1697	.9890	9952	.1495	1745	6.6912	8255	30
40	.1507	1781	.9886	9950	.1524	1831	6.5606	8169	20
50	.1536	1863	.9881	9948	.1554	1915	6.4348	8085	10
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	81° 00′
10	.1593	2022	.9872	9944	.1614	2078	6.1970	7922	50
20	.1622	2100	.9868	9942	.1644	2158	6.0844	7842	40
30	.1650	2176	.9863	9940	.1673	2236	5.9758	7764	30
40	.1679	2251	.9858	9938	.1703	2313	5.8707	7687	20
50	.1708	2324	.9853	9936	.1733	2389	5.7694	7611	10
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	80° 00′
10	.1765	2468	.9843	9931	.1793	2536	5.5764	7464	50
20	.1794	2538	.9838	9929	.1823	2609	5.4845	7391	40
30	.1822	2606	.9833	9927	.1853	2680	5.3955	7320	30
40	.1851	2674	.9827	9924	.1883	2750	5.3093	7250	20
50	.1880	2740	.9822	9922	.1914	2819	5.2257	7181	10
11° 00′	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	79° 00′
10	.1937	2870	.9811	9917	.1974	2953	5.0658	7047	50
20	.1965	2934	.9805	9914	.2004	3020	4.9894	6980	40
30	.1994	2997	.9799	9912	.2035	3085	4.9152	6915	30
40	.2022	3058	.9793	9909	.2065	3149	4.8430	6851	20
50	.2051	3119	.9787	9907	.2095	3212	4.7729	6788	10
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	78° 00′
10	.2108	3238	.9775	9901	.2156	3336	4.6382	6664	50
20	.2136	3296	.9769	9899	.2186	3397	4.5736	6603	40
30	.2164	3353	.9763	9896	.2217	3458	4.5107	6542	30
40	.2193	3410	.9757	9893	.2247	3517	4.4494	6483	20
50	.2221	3466	.9750	9890	.2278	3576	4.3897	6424	10
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	77° 00′
10	.2278	3575	.9737	9884	.2339	3691	4.2747	6309	50
20	.2306	3629	.9730	9881	.2370	3748	4.2193	6252	40
30	.2334	3682	.9724	9878	.2401	3804	4.1653	6196	30
40	.2363	3734	.9717	9875	.2432	3859	4.1126	6141	20
50	.2391	3786	.9710	9872	.2462	3914	4.0611	6086	10
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	76° 00′
10	.2447	3887	.9696	9866	.2524	4021	3.9617	5979	50
20	.2476	3937	.9689	9863	.2555	4074	3.9136	5926	40
30	.2504	3986	.9681	9859	.2586	4127	3.8667	5873	30
40	.2532	4035	.9674	9856	.2617	4178	3.8208	5822	20
50	.2560	4083	.9667	9853	.2648	4230	3.7760	5770	10
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	75° 00′
10	.2616	4177	.9652	9846	.2711	4331	3.6891	5669	50
20	.2644	4223	.9644	9843	.2742	4381	3.6470	5619	40
30	.2672	4269	.9636	9839	.2773	4130	3.6059	5570	30
40	.2700	4314	.9628	9836	.2805	4479	3.5656	5521	20
50	.2728	4359	.9621	9832	.2836	4527	3.5261	5473	10
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	74° 00′
10	.2784	4447	.9605	9825	.2899	4622	3.4495	5378	50
20	.2812	4491	.9596	9821	.2931	4669	3.4124	5331	40
30	.2840	4533	.9588	9817	.2962	4716	3.3759	5284	30
40	.2868	4576	.9580	9814	.2994	4762	3.3402	5238	20
50	.2896	4618	.9572	9810	.3026	4808	3.3052	5192	10
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	73° 00′
10	.2952	4700	.9555	9802	.3089	4898	3.2371	5102	50
20	.2979	4741	.9546	9798	.3121	4943	3.2041	5057	40
30	.3007	4781	.9537	9794	.3153	4987	3.1716	5013	30
40	.3035	4821	.9528	9790	.3185	5031	3.1397	4969	20
50	.3062	4861	.9520	9786	.3217	5075	3.1084	4925	10
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	72° 00′
10	.3118	4939	.9502	9778	.3281	5161	3.0475	4839	50
20	.3145	4977	.9492	9774	.3314	5203	3.0178	4797	40
30	.3173	5015	.9483	9770	.3346	5245	2.9887	4755	30
40	.3201	5052	.9474	9765	.3378	5287	2.9600	4713	20
50	.3228	5090	.9465	9761	.3411	5329	2.9319	4671	10
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	71° 00′
10	.3283	5163	.9446	9752	.3476	5411	2.8770	4589	50
20	.3311	5199	.9436	9748	.3508	5451	2.8502	4549	40
30	.3338	5235	.9426	9743	.3541	5491	2.8239	4509	30
40	.3365	5270	.9417	9739	.3574	5531	2.7980	4469	20
50	.3393	5306	.9407	9734	.3607	5571	2.7725	4429	10
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	70° 00′
10	.3448	5375	.9387	9725	.3673	5650	2.7228	4350	50
20	.3475	5409	.9377	9721	.3706	5689	2.6985	4311	40
30	.3502	5443	.9367	9716	.3739	5727	2.6746	4273	30
40	.3529	5477	.9356	9711	.3772	5766	2.6511	4234	20
50	.3557	5510	.9346	9706	.3805	5804	2.6279	4196	10
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	69° 00′
10	.3611	5576	.9325	9697	.3872	5879	2.5826	4121	50
20	.3638	5609	.9315	9692	.3906	5917	2.5605	4083	40
30	.3665	5641	.9304	9687	.3939	5954	2.5386	4046	30
40	.3692	5673	.9293	9682	.3973	5991	2.5172	4009	20
50	.3719	5704	.9283	9677	.4006	6028	2.4960	3972	10
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	68° 00′
10	.3773	5767	.9261	9667	.4074	6100	2.4545	3900	50
20	.3800	5798	.9250	9661	.4108	6136	2.4342	3864	40
30	.3827	5828	.9239	9656	.4142	6172	2.4142	3828	30
40	.3854	5859	.9228	9651	.4176	6208	2.3945	3792	20
50	.3881	5889	.9216	9646	.4210	6243	2.3750	3757	10
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	67° 00′
10	.3934	5948	.9194	9635	.4279	6314	2.3369	3686	50
20	.3961	5978	.9182	9629	.4314	6348	2.3183	3652	40
30	.3987	6007	.9171	9624	.4348	6383	2.2998	3617	30
40	.4014	6036	.9159	9618	.4383	6417	2.2817	3583	20
50	.4041	6065	.9147	9613	.4417	6452	2.2637	3548	10
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	66° 00′
10	.4094	6121	.9124	9602	.4487	6520	2.2286	3480	50
20	.4120	6149	.9122	9596	.4522	6553	2.2113	3447	40
30	.4147	6177	.9100	9590	.4557	6587	2.1943	3413	30
40	.4173	6205	.9088	9584	.4592	6620	2.1775	3380	20
50	.4200	6232	.9075	9579	.4628	6654	2.1609	3346	10
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	65° 00′
10	.4253	6286	.9051	9567	.4699	6720	2.1283	3280	50
20	.4279	6313	.9038	9561	.4734	6752	2.1123	3248	40
30	.4305	6340	.9026	9555	.4770	6785	2.0965	3215	30
40	.4331	6366	.9013	9549	.4806	6817	2.0809	3183	20
50	.4358	6392	.9001	9543	.4841	6850	2.0655	3150	10
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	64° 00′
10	.4410	6444	.8975	9530	.4913	6914	2.0353	3086	50
20	.4436	6470	.8962	9524	.4950	6946	2.0204	3054	40
30	.4462	6495	.8949	9518	.4986	6977	2.0057	3023	30
40	.4488	6521	.8936	9512	.5022	7009	1.9912	2991	20
50	.4514	6546	.8923	9505	.5059	7040	1.9768	2960	10
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

TABLE II.—TRIGONOMETRIC FUNCTIONS—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
27° 00	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	63° 00′
10	.4566	6595	.8897	9492	.5132	7103	1.9486	2897	50
20	.4592	6620	.8884	9486	.5169	7134	1.9347	2866	40
30	.4617	6644	.8870	9479	.5206	7165	1.9210	2835	30
40	.4643	6668	.8857	9473	.5243	7196	1.9074	2804	20
50	.4669	6692	.8843	9466	.5280	7226	1.8940	2774	10
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	62° 00′
10	.4720	6740	.8816	9453	.5354	7287	1.8676	2713	50
20	.4746	6763	.8802	9446	.5392	7317	1.8546	2683	40
30	.4772	6787	.8788	9439	.5430	7348	1.8418	2652	30
40	.4797	6810	.8774	9432	.5467	7378	1.8291	2622	20
50	.4823	6833	.8760	9425	.5505	7408	1.8165	2592	10
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	61° 00′
10	.4874	6878	.8732	9411	.5581	7467	1.7917	2533	50
20	.4899	6901	.8718	9404	.5619	7497	1.7796	2503	40
30	.4924	6923	.8704	9397	.5658	7526	1.7675	2474	30
40	.4950	6946	.8689	9390	.5696	7556	1.7556	2444	20
50	.4975	6968	.8675	9383	.5735	7585	1.7437	2415	10
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	60° 00′
10	.5025	7012	.8646	9368	.5812	7644	1.7205	2356	50
20	.5050	7033	.8631	9361	.5851	7673	1.7090	2327	40
30	.5075	7055	.8616	9353	.5890	7701	1.6977	2299	30
40	.5100	7076	.8601	9346	.5930	7730	1.6864	2270	20
50	.5125	7097	.8587	9338	.5969	7759	1.6753	2241	10
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	59° 00′
10	.5175	7139	.8557	9323	.6048	7816	1.6534	2184	50
20	.5200	7160	.8542	9315	.6088	7845	1.6426	2155	40
30	.5225	7181	.8526	9308	.6128	7873	1.6319	2127	30
40	.5250	7201	.8511	9300	.6168	7902	1.6212	2098	20
50	.5275	7222	.8496	9292	.6208	7930	1.6107	2070	10
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	58° 00′
10	.5324	7262	.8465	9276	.6289	7986	1.5900	2014	50
20	.5348	7282	.8450	9268	.6330	8014	1.5798	1986	40
30	.5373	7302	.8434	9260	.6371	8042	1.5697	1958	30
40	.5398	7322	.8418	9252	.6412	8070	1.5597	1930	20
50	.5422	7342	.8403	9244	.6453	8097	1.5497	1903	10
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	57° 00′
10	.5471	7380	.8371	9228	.6536	8153	1.5301	1847	50
20	.5495	7400	.8355	9219	.6577	8180	1.5204	1820	40
30	.5519	7419	.8339	9211	.6619	8208	1.5108	1792	30
40	.5544	7438	.8323	9203	.6661	8235	1.5013	1765	20
50	.5568	7457	.8307	9194	.6703	8263	1.4919	1737	10
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	56° 00′
10	.5616	7494	.8274	9177	.6787	8317	1.4733	1683	50
20	.5640	7513	.8258	9169	.6830	8344	1.4641	1656	40
30	.5664	7531	.8241	9160	.6873	8371	1.4550	1629	30
40	.5688	7550	.8225	9151	.6916	8398	1.4460	1602	20
50	.5712	7568	.8208	9142	.6959	8425	1.4370	1575	10
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	55° 00′
10	.5760	7604	.8175	9125	.7046	8479	1.4193	1521	50
20	.5783	7622	.8158	9116	.7089	8506	1.4106	1494	40
30	.5807	7640	.8141	9107	.7133	8533	1.4019	1467	30
40	.5831	7657	.8124	9098	.7177	8559	1.3934	1441	20
50	.5854	7675	.8107	9089	.7221	8586	1.3848	1414	10
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

Table II.—Trigonometric Functions—*Continued*

Angles	Sines		Cosines		Tangents		Cotangents		Angles
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	54° 00′
10	.5901	7710	.8073	9070	.7310	8639	1.3680	1361	50
20	.5925	7727	.8056	9061	.7355	8666	1.3597	1334	40
30	.5948	7744	.8039	9052	.7400	8692	1.3514	1308	30
40	.5972	7761	.8021	9042	.7445	8718	1.3432	1282	20
50	.5995	7778	.8004	9033	.7490	8745	1.3351	1255	10
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	53° 00′
10	.6041	7811	.7969	9014	.7581	8797	1.3190	1203	50
20	.6065	7828	.7951	9004	.7627	8824	1.3111	1176	40
30	.6088	7844	.7934	8995	.7673	8850	1.3032	1150	30
40	.6111	7861	.7916	8985	.7720	8876	1.2954	1124	20
50	.6134	7877	.7898	8975	.7766	8902	1.2876	1098	10
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2799	0.1072	52° 00′
10	.6180	7910	.7862	8955	.7860	8954	1.2723	1046	50
20	.6202	7926	.7844	8945	.7907	8980	1.2647	1020	40
30	.6225	7941	.7826	8935	.7954	9006	1.2572	0994	30
40	.6248	7957	.7808	8925	.8002	9032	1.2497	0968	20
50	.6271	7973	.7790	8915	.8050	9058	1.2423	0942	10
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	51° 00′
10	.6316	8004	.7753	8895	.8146	9110	1.2276	0890	50
20	.6338	8020	.7735	8884	.8195	9135	1.2203	0865	40
30	.6361	8035	.7716	8874	.8243	9161	1.2131	0839	30
40	.6383	8050	.7698	8864	.8292	9187	1.2059	0813	20
50	.6406	8066	.7679	8853	.8342	9212	1.1988	0788	10
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	50° 00′
10	.6450	8096	.7642	8832	.8441	9264	1.1847	0736	50
20	.6472	8111	.7623	8821	.8491	9289	1.1778	0711	40
30	.6494	8125	.7604	8810	.8541	9315	1.1708	0685	30
40	.6517	8140	.7585	8800	.8591	9341	1.1640	0659	20
50	.6539	8155	.7566	8789	.8642	9366	1.1571	0634	10
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	49° 00′
10	.6583	8184	.7528	8767	.8744	9417	1.1436	0583	50
20	.6604	8198	.7509	8756	.8796	9443	1.1369	0557	40
30	.6626	8213	.7490	8745	.8847	9468	1.1303	0532	30
40	.6648	8227	.7470	8733	.8899	9494	1.1237	0506	20
50	.6670	8241	.7451	8722	.8952	9519	1.1171	0481	10
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	48° 00′
10	.6713	8269	.7412	8699	.9057	9570	1.1041	0430	50
20	.6734	8283	.7392	8688	.9110	9595	1.0977	0405	40
30	.6756	8297	.7373	8676	.9163	9621	1.0913	0379	30
40	.6777	8311	.7353	8665	.9217	9646	1.0850	0354	20
50	.6799	8324	.7333	8653	.9271	9671	1.0786	0329	10
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	47° 00′
10	.6841	8351	.7294	8629	.9380	9722	1.0661	0278	50
20	.6862	8365	.7274	8618	.9435	9747	1.0599	0253	40
30	.6884	8378	.7254	8606	.9490	9772	1.0538	0228	30
40	.6905	8391	.7234	8594	.9545	9798	1.0477	0202	20
50	.6926	8405	.7214	8582	.9601	9823	1.0416	0177	10
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	46° 00′
10	.6967	8431	.7173	8557	.9713	9874	1.0295	0126	50
20	.6988	8444	.7153	8545	.9770	9899	1.0235	0101	40
30	.7009	8457	.7133	8532	.9827	9924	1.0176	0076	30
40	.7030	8469	.7112	8520	.9884	9949	1.0117	0051	20
50	.7050	8482	.7092	8507	.9942	9975	1.0058	0025	10
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	45° 00′
	Nat.	Log.	Nat.	Log.	Nat.	Log.	Nat.	Log.	
Angles	Cosines		Sines		Cotangents		Tangents		Angles

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ANSWERS

Exercises 1-1, page 6

		(a)	(b)	(c)	(d)	(e)	(f)
1.	tan A	$\frac{3}{2}$	1	$\frac{4}{3}$	$\frac{1}{4}$	10	$\frac{15}{8}$
	tan B	$\frac{2}{3}$	1	$\frac{3}{4}$	4	$\frac{1}{10}$	$\frac{8}{15}$

		(a)	(b)	(c)	(d)
2.	sin A	$\frac{3}{5}$	$\frac{4}{5}$	$\dfrac{3}{\sqrt{10}}$	$\dfrac{1}{\sqrt{101}}$
	cos A	$\frac{4}{5}$	$\frac{3}{5}$	$\dfrac{1}{\sqrt{10}}$	$\dfrac{10}{\sqrt{101}}$
	tan A	$\frac{3}{4}$	$\frac{4}{3}$	3	$\frac{1}{10}$

3. $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$
4. $\sin A = \frac{24}{25}$, $\tan A = \frac{24}{7}$
5. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$
6. $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$
7. $\sin A = \frac{7}{25}$, $\tan A = \frac{7}{24}$
8. $\sin A = \frac{8}{17}$, $\tan A = \frac{8}{15}$
11. 550 ft. **12.** 1120 ft. **13.** 9 ft. **14.** 198.5 ft. **15.** 1500 ft.

Exercises 1-2, page 9

		(a)	(b)	(c)	(d)	(e)	(f)
1.	sin A	$\dfrac{3}{\sqrt{34}}$	$\frac{3}{5}$	$\frac{3}{5}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{\sqrt{5}}$	$\frac{21}{29}$
	cos A	$\dfrac{5}{\sqrt{34}}$	$\frac{4}{5}$	$\frac{4}{5}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{2}{\sqrt{5}}$	$\frac{20}{29}$
	tan A	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{21}{20}$

2. $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}, \frac{12}{5}, \frac{13}{12}, \frac{13}{5}; \frac{12}{13}, \frac{5}{13}, \frac{12}{5}, \frac{5}{12}, \frac{13}{5}, \frac{13}{12}$

3. (a) $\cos \theta = \frac{3}{5}$, $\tan \theta = \frac{4}{3}$, $\cot \theta = \frac{3}{4}$, $\sec \theta = \frac{5}{3}$, $\csc \theta = \frac{5}{4}$

(b) $\sin \theta = \frac{8}{17}$, $\cos \theta = \frac{15}{17}$, $\cot \theta = \frac{15}{8}$, $\sec \theta = \frac{17}{15}$, $\csc \theta = \frac{17}{8}$

(c) $\sin \theta = \dfrac{\sqrt{3}}{2}$, $\tan \theta = \sqrt{3}$, $\cot \theta = \dfrac{1}{\sqrt{3}}$, $\sec \theta = 2$, $\csc \theta = \dfrac{2}{\sqrt{3}}$

4. (a) 1; (b) 1 **6.** 180 ft. **7.** 45 ft.
8. 396 ft. **9.** $a = 738.5$ ft.; $b = 307.7$ ft.

10. (a) $a = 312$, $b = 416$ (b) $c = 997.3$, $b = 469.3$ (c) $a = 68$, $c = 34\sqrt{5}$

(d) $a = 230.8$, $b = 96.1$ (e) $b = 17.3\sqrt{10}$, $c = 51.9\sqrt{10}$ (f) $a = 284\sqrt{10}$, $c = 852$

Exercises 1-3, page 12

2. 0.000291, 1, 0.000291, etc.; 1, 0.000291, 3436, etc.

3. $\cos A = \frac{40}{41}$, $\sin(90° - A) = \frac{40}{41}$

$\tan A = \frac{9}{40}$, $\cos(90° - A) = \frac{9}{41}$

$\cot A = \frac{40}{9}$, $\tan(90° - A) = \frac{40}{9}$

$\sec A = \frac{41}{40}$, $\cot(90° - A) = \frac{9}{40}$

$\csc A = \frac{41}{9}$, $\sec(90° - A) = \frac{41}{9}$

$\csc(90° - A) = \frac{41}{40}$

4. $\dfrac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}, \dfrac{1}{\sqrt{3}}, 2, \dfrac{2}{\sqrt{3}}$

5. $\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 1, 1, \sqrt{2}, \sqrt{2}$

8. (a) $\dfrac{1}{\sqrt{3}}$; (b) $\sqrt{6}$; (c) 1; (d) $\dfrac{1}{3\sqrt{2}}$

10. $\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}, \frac{3}{2}, \frac{2}{3}, \dfrac{\sqrt{13}}{2}, \dfrac{\sqrt{13}}{3}$

11. $\dfrac{1}{2\sqrt{2}}(\sqrt{3} + 1), \dfrac{\sqrt{12}}{2(\sqrt{3} - 1)}$

12. 0.577 miles

13. 22.5 ft.

14. 482.3 yd.

Exercises 1-4, page 13

1. (a) $\dfrac{2}{\sqrt{29}}, \dfrac{5}{\sqrt{29}}, \frac{2}{5}, \frac{5}{2}, \dfrac{\sqrt{29}}{5}, \dfrac{\sqrt{29}}{2}$; (b) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}$ **2.** $\frac{15}{17}, \frac{8}{17}, \frac{8}{15}$

4. (a) $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, $\cot A = \frac{3}{4}$, $\sec A = \frac{5}{3}$, $\csc A = \frac{5}{4}$

(b) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\cot A = \frac{15}{8}$, $\sec A = \frac{17}{15}$, $\csc A = \frac{17}{8}$

(c) $\sin A = \frac{5}{13}$, $\tan A = \frac{5}{12}$, $\cot A = \frac{12}{5}$, $\sec A = \frac{13}{12}$, $\csc A = \frac{13}{5}$

5. (a) $\frac{336}{625}$; (b) $-\frac{527}{625}$ **6.** 1 **7.** $\frac{1}{8}(3 + \sqrt{21})$ **8.** 39, 36

9. $b = 65$, $c = 57$, $a = 68$; Altitude to $b = 52.62$; altitude to $a = 50.34$

10. 75, 125, $\dfrac{750}{\sqrt{61}}, \dfrac{625}{\sqrt{61}}$ **11.** $a = 12$, $b = 6\sqrt{3}$, $c = 3\sqrt{6}$

12. $x = 13.5$, $y = 19.7$, $z = 22.5$ **13.** $x = 19.2$, $y = 14.4$, $z = 10$

14. $s = 6$, $t = 5.54$, $w = 2.31$, $x = 8$, $y = 3.08$, $z = 7.38$

15. $x = 150$, $w = 250$, $y = 117.6$, $z = 220.6$

16. $a = 3\sqrt{34}$, $b = 4\sqrt{34}$, $c = 5\sqrt{34}$; $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{4}, \frac{5}{3}$

17. $AD = 28$, $\sin\beta = \frac{20}{29}$, $\sin\gamma = \frac{3}{5}$, $\sin\delta = \dfrac{7}{\sqrt{130}}$

$AO = 21$, $\cos\beta = \frac{21}{29}$, $\cos\gamma = \frac{4}{5}$, $\cos\delta = \dfrac{9}{\sqrt{130}}$

$OB = 20,\quad \tan\beta = \frac{20}{21},\ \tan\gamma = \frac{3}{4},\ \cos\delta = \frac{7}{9}$

$OC = 15,\quad \cot\beta = \frac{21}{20},\ \cot\gamma = \frac{4}{3},\ \cot\delta = \frac{9}{7}$

$DC = 4\sqrt{130},\quad \sec\beta = \frac{29}{21},\ \sec\gamma = \frac{5}{4},\ \sec\delta = \frac{\sqrt{130}}{9}$

$OE = \frac{21}{26}\sqrt{130},\ \csc\beta = \frac{29}{20},\ \csc\gamma = \frac{5}{3},\ \csc\delta = \frac{\sqrt{130}}{7}$

18. $AO = 57.12$ ft.

19. $CD = 12,\quad \sin DEC = \frac{5}{\sqrt{34}}$

$AD = 35,\quad \cos DEC = \frac{3}{\sqrt{34}}$

$AB = 30,\quad \tan DEC = \frac{5}{3}$

$AE = EB = 15,\ \cot DEC = \frac{3}{5}$

$CB = 13,\quad \sec DEC = \frac{\sqrt{34}}{3}$

$CE = 4\sqrt{34},\quad \csc DEC = \frac{\sqrt{34}}{5}$

20. $AD = 25,\ \sin AED = \frac{15}{\sqrt{481}}$

$DB = 15,\ \cos AED = \frac{16}{\sqrt{481}}$

$AE = \frac{80}{3},\ \tan AED = \frac{15}{16}$

$CE = \frac{64}{3}$

$ED = \frac{5}{3}\sqrt{481}$

21. $DA = 1,\ DC = 1,\ OD = \sqrt{3},\ DB = 2 - \sqrt{3},\ AB = 2\sqrt{2 - \sqrt{3}};$

$\sin 15° = \frac{\sqrt{2 - \sqrt{3}}}{2},\ \cos 15° = \frac{1}{2\sqrt{2 - \sqrt{3}}},\ \tan 15° = 2 - \sqrt{3}$

22. $\sin 22\frac{1}{2}° = \frac{\sqrt{2 - \sqrt{2}}}{2},\ \cos 22\frac{1}{2}° = \frac{\sqrt{2}}{2\sqrt{2 - \sqrt{2}}},\ \tan 22\frac{1}{2}° = \frac{2 - \sqrt{2}}{\sqrt{2}}$

24. $y = 74.27$ **25.** 72.14 **26.** 16,500 ft. **27.** 2 miles

Exercises 2-1, page 19

2. (*a*) 0.213; (*b*) 1.252; (*c*) 0.213; (*d*) 0.788; (*e*) 0.988; (*f*) 0.485; (*g*) 1.192; (*h*) 0.819; (*i*) 0.445

4. (*a*) 24°; (*b*) 33°; (*c*) 15°; (*d*) 9v; (*e*) 62°; (*f*) 68°; (*g*) 42°; (*h*) 21°; (*i*) 54°

Exercises 2-2, page 21

1. (*a*) $a = 42$, $b = 50$ (*b*) $b = 63$, $c = 98$ (*c*) $a = 141$, $c = 812$ (*d*) $a = 96$, $c = 102$

2. (*a*) $a = 49$, $b = 70$ (*b*) $b = 1134$, $c = 1152$ (*c*) $b = 350$, $c = 611$ (*d*) $b = 19$, $a = 5$ (*e*) $a = 42$, $b = 91$ (*f*) $a = 22$, $c = 64$

3. 68 ft. **4.** 246 ft., 172 ft. **5.** 275 ft. **6.** 66 ft.

7. 38 ft. **8.** 58.4 ft. **9.** 5590 ft. **10.** 105 ft.

11. 3.9 ft. **12.** 24.28 yd. **13.** 2427.6 ft. **14.** 53,631 ft.

15. 57.12 ft. **16.** 2.76 cm. **18.** 5272 ft. **19.** 184.6 ft. **20.** 16.78 miles

21. $v = 2.4,\ w = 3.2,\ q = 5.52,\ R = 2.330,\ s = 2.517,\ t = 3.915$

Exercises 2-3, page 26

1. (*a*) 6.72; (*b*) 985; (*c*) 69,300; (*d*) 4940 **2.** 49 ft.

Exercises 2-4, page 27

1. 0.678 **2.** 0.582 **3.** 0.407 **4.** 2.663 **5.** 2.153
6. 3.563 **7.** 0.209 **8.** 0.965 **9.** 2.005 **10.** 0.700
11. 0.289 **12.** 0.845 **13.** 42°13′ **14.** 24°46′ **15.** 58°28′
16. 62°37′ **17.** 33°34′ **18.** 17°27′ **19.** 57°13′ **20.** 43°40′
21. 25°44′
22. $b = 28.40$, $c = 42.78$, $B = 41°35'$
23. $a = 40.23$, $b = 22.52$, $A = 60°46'$
24. $A = 40°30'$, $B = 49°30'$, $b = 19.1$
25. $A = 50°27'$, $B = 39°33'$, $c = 3.94$
26. $a = 106.2$, $c = 125.6$, $A = 57°45'$
27. $a = 22.2$, $b = 42.1$, $B = 27°48'$
28. $c = 45.6$, $A = 64°0'$, $B = 26°0'$
29. $a = 12.8$, $b = 34.7$, $B = 20°10'$

Exercises 2-5, page 29

1. 8°5′ **2.** 6.30 miles, 8.04 miles **3.** 0.72 mile **4.** 114.3 ft.
5. 50°33′ **6.** 11.48 ft. **7.** 6281 ft. **8.** 3214 ft.
9. 99.0 ft. **10.** 20.90 ft. **11.** 0.130 mile

Exercises 2-6, page 33

1. $A = 36°52'$, $B = 53°8'$, $b = 80$
2. $B = 26°$, $a = 410$, $c = 457$
3. $A = 27°4'$, $a = 24.37$, $c = 53.56$
4. $B = 51°20'$, $c = 80.9$, $b = 63.2$
5. $A = 83°48'$, $a = 36.98$, $b = 4.02$
6. $A = 43°18'$, $B = 46°42'$, $b = 0.662$
7. $A = 21°10'$, $b = 1884$, $c = 2020$
8. $B = 46°30'$, $a = 7.71$, $b = 8.12$
9. $B = 17°53'$, $b = 26.91$, $c = 87.6$

Exercises 2-7, page 34

1. $A = 31°20'$, $B = 58°40'$, $c = 237$
2. $A = 33°9'$, $B = 56°51'$, $c = 499$
3. $A = 45°0'$, $B = 45°0'$, $c = 18.67$
4. $A = 41°2'$, $B = 48°58'$, $c = 153.8$
5. $A = 39°30'$, $B = 50°30'$, $c = 44$
6. $A = 30°37'$, $B = 59°23'$, $c = 82.5$
7. $A = 65°0'$, $B = 25°0'$, $c = 55.2$
8. $A = 67°23'$, $B = 22°37'$, $c = 13$
9. $A = 3°42'$, $B = 86°18'$, $c = 4.8$

Exercises 2-8, page 37

1. 9.8060 − 10 2. 9.9354 − 10 3. 9.1777 − 10 4. 9.7345 − 10
5. 9.9351 − 10 6. 9.9565 − 10 7. 9.5654 − 10 8. 9.9822 − 10
9. 9.9950 − 10 10. 9.9899 − 10

Exercises 2-9, page 38

1. 11°55′ 2. 6°8′ 3. 44°12′ 4. 7°44′ 5. 33°26′
6. 80°32′ 7. 52°13′ 8. 53°58′ 9. 6°2′ 10. 5°12′

Exercises 2-10, page 39

1. $a = 9.8$, $c = 17$, $B = 55°0'$
2. $a = 5.941$, $b = 2.021$, $A = 71°13'$
3. $b = 811.5$, $A = 47°30'$, $B = 42°30'$
4. $c = 9.02$, $A = 74°9'$, $B = 15°51'$
5. $a = 388.3$, $b = 549$, $B = 54°44'$
6. $c = 757.2$, $A = 58°27'$, $B = 31°33'$
7. $b = 22.66$, $A = 76°40'$, $B = 13°20'$
8. $b = 18.16$, $c = 39.8$, $A = 62°51'$
9. $a = 17.34$, $b = 17.85$, $B = 45°50'$
10. $c = 6.656$, $A = 29°38'$, $B = 60°22'$
11. $b = 17.60$, $c = 74.25$, $B = 13°43'$
12. $a = 193.6$, $b = 1661$, $A = 6°39'$
13. 30.56 ft. 14. 65.71 miles 15. 2964 ft.
16. 0°20′ 17. 9.88 ft. 18. 35°16′
19. 18.6 in. 20. 10,524 ft. 21. 35°32′
22. 957.8 ft. 23. 100 ft. 24. 2957.2 miles
25. 1°9′, 8100 ft. 26. 8.17, 7.55 27. 40°47′

Exercises 2-11, page 42

1. 48.80 ft. 2. 14.40 ft. 3. $BC = m \tan^2 A$, $DE = \dfrac{m \tan^2 A}{\cos A}$
4. $MN = a \cot \phi \cos^2 \phi$ $CE = \dfrac{m \tan A}{\cos^2 A}$
5. $AOB = 11.10$ 6. $x = m \sin (\theta - \varphi) \csc (\theta - \alpha) \cos \alpha$
7. 133.7 8. 4470 ft. 9. 89.3 ft. 10. 272.4 ft.
11. 864 ft., 708 ft., 246 ft. 12. 69.77 ft. 13. 275.9 ft.
14. (*a*) 20.56 miles; (*b*) 39.85 miles

Exercises 2-12, page 46

1. 127.2 miles, 141.2 miles
2. 22°20.5′, 78.57 miles
3. 24°55.9′, 65.66 miles
4. (*a*) 176 miles, 94 miles
 (*b*) 90 miles, 120 miles
 (*c*) 285 miles, 93 miles
 (*d*) 192 miles, 161 miles
5. 179.6 miles
6. 32.4 miles, 120.7 miles
7. 464.0 ft.
8. 339°26.5′, 10.62 knots

9. 5.45 miles, 25′3″
10. 3.91 miles
11. 1 hr. 16.3 min., 9.64 miles
12. 2.73 miles, 14 min. 24 sec.

Exercises 2-13, page 50

1. (*a*) 20.48, 14.34
(*b*) 94.37, 46.03
(*c*) 9.37, 10.80
(*d*) 11.40, 17.18
2. 129.32, 178.0
3. 21°48′, 10.8 miles per hour
4. 855 lb., 2349 lb.
5. 10°37′, 16.3 miles per hour
6. 54°27′, 17.2 lb.
7. 380 lb., 124 lb.
8. 1230.6 lb.
9. 163.8 lb., 114.7 lb.
10. 771.4 lb.
11. 159.7 miles, 80°35′
12. 50 miles, 98°19′
13. 262°31′, 12.57 miles

Exercises 2-14, page 51

1. $A = 34°12'$, $b = 153.0$, $B = 55°48'$
2. $a = 58.24$, $c = 75.33$, $A = 50°38'$
3. $A = 22°52'$, $b = 5.428$, $a = 2.289$
4. $a = 434.2$, $b = 449.6$, $B = 46°$
5. $A = 27°16'$, $b = 9694$, $c = 10{,}907.5$
6. $B = 63°12'$, $A = 26°48'$, $c = 8.878$
7. 5178.8 yd.
8. 4880 cu. yd.
9. Radius $= \frac{9}{32}(3\sqrt{2} - 2\sqrt{3})$
10. 139°10′, 80.60 miles
11. 0.714 miles
12. 24,099 sq. ft.
13. 34.15 ft.
14. 142.5 ft., 128 ft.
15. (*a*) 3.42 miles; (*b*) 6.83 miles
16. 28°23′
17. 10,910 ft.
18. 345.8 ft., 116.8 ft.
19. 284 ft., 291 ft.
20. 7.87 miles
27. 1000 ft.
28. 1839 ft.
29. 43.34 miles
30. 1°47′
31. 5713 ft.
32. 348°28′
33. 350.5 ft.
34. 2°8′
35. 3.92 miles
36. 2.89 miles
37. 123 ft.
38. 15.43′
39. 328°36′
40. 1.62 miles

Exercises 3-1, page 61

1. (*a*) cos 15°; (*b*) sin 3°; (*c*) cot 30′;
(*d*) csc 40°40′; (*e*) tan 44°10′; (*f*) sec 19°20′
2. 20°, 10°, 5°, 9°20′
3. (*a*) $\cos\theta$; (*b*) $\sin\theta$; (*c*) $\csc\theta$; (*d*) 1; (*e*) $\sec\theta$;
(*f*) $\cos\theta$; (*g*) 1; (*h*) 1; (*i*) $\tan\theta$
6. 11°51.4′, 6°28′, 4°35.3′, 14°42′

Exercises 3-2, page 63

1. (*a*) $\cos^2\beta$; (*b*) $\sin^2\beta$; (*c*) $\tan^2\beta$; (*d*) 1; (*e*) $-\cot^2\beta$; (*f*) 1; (*g*) 1;
(*h*) $-\sin^2\theta\tan^2\theta$
2. (*a*) 1; (*b*) 1; (*c*) $\cot^2\varphi$; (*d*) $\dfrac{1}{\sin\varphi\cos\varphi}$; (*e*) 1; (*f*) 1
3. (*a*) $2\sin^3\theta - 2\sin^5\theta$; (*b*) $2\sin^2\theta - 1$; (*c*) $1 - 2\sin^2\theta$; (*d*) $2\sin^2\theta - 1$

Exercises 3-3, page 66

1. $\sec x$
2. $\tan A$
3. 1
4. 1
5. -1
6. -1

Exercises 3-5, page 72

2. $DE = a \cos A \sec B$, $CE = a \sin A + a \cos A \tan B$
3. $a \cos^4 \theta$ **4.** $a \sin^4 \theta$ **5.** 43.2, 75.23 **6.** 71.88, 92.21
7. $\tan \frac{1}{2}\theta = \dfrac{\sin \theta}{1 + \cos \theta}$ **8.** $AB = a \sin^2 \theta$, $DE = a \cos^2 \theta$
9. $FD = \sin \varphi \sin \theta$, $CD = \cos \varphi \sin \theta$
10. $FD = \sec \theta \tan \varphi \sin \theta = \tan \theta \tan \varphi$
11. $\sin 2\theta = 2 \sin \theta \cos \theta$

Exercises 3-6, page 74

1. (*a*) $\cos 25°$; (*b*) $\cot 51°$; (*c*) $\csc 8°$
2. (*a*) $\cos^2 \theta$; (*b*) 1; (*c*) 2; (*d*) $\sec^2 \theta$; (*e*) $\sec^2 \theta$; (*f*) $\sin^2 \theta$; (*g*) 2
4. (*a*) $\dfrac{1 - \sin^2 A}{\sin A}$; (*b*) $\dfrac{1}{\sin A}$; (*c*) $\sin A$; (*d*) $1 - 2 \sin^2 A$
5. (*a*) $\cos A$; (*b*) $\cos^2 A$ **6.** (*a*) $\tan \theta$; (*b*) $\tan^2 \theta + \tan^4 \theta$
7. (*a*) $\dfrac{1}{\sin \theta \cos \theta}$; (*b*) $\dfrac{1 - \cos \theta}{\sin \theta}$; (*c*) $\dfrac{1 + \sin \theta}{\cos \theta}$
9. (*a*) $a \sin \theta$; (*b*) $b \sin \theta$; (*c*) $b \tan 6$; (*d*) $a \sin^4 \theta$;
(*e*) $a \sin^6 \theta$; (*f*) $b \csc \theta$; (*g*) $b \sin \theta \sec \theta$;
(*h*) $2a \sin^3 \theta \sec \theta$; (*i*) $2a \cos \theta$
31. 12.68 **32.** 69.14, 107.5 **33.** 7.0, 12.2
34. $x = 13.004$, $y = 21.79$ **35.** $AC = a \sin \theta \cot \phi$, $AB = a \sin \theta \cot \phi \cot \alpha$

Exercises 4-1, page 79

2. 7 **3.** $\frac{5}{3}$ right angles clockwise **4.** $\frac{1}{15}$
5. 24 **6.** (*a*) 1; (*b*) $2\frac{1}{3}$; (*c*) $8\frac{1}{3}$; (*d*) 8000; (*e*) $\frac{4}{365}$; (*f*) $\frac{1}{2190}$

Exercises 4-2, page 81

3. (*a*) $\frac{5}{13}$, $\frac{12}{13}$, $\frac{5}{12}$, etc.; (*b*) $\dfrac{y}{\sqrt{x^2 + y^2}}$, $\dfrac{x}{\sqrt{x^2 + y^2}}$, $\dfrac{y}{x}$, etc.
5. 0, 0 **6.** (*a*) I; (*b*) II; (*c*) IV; (*d*) III
7.

	(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)
Pos.	I, II	I, IV	I, III	I, III	I, IV	I, II
Neg.	III, IV	II, III	II, IV	II, IV	II, III	III, IV

Exercises 4-3, page 82

1. (*a*) $-\frac{4}{5}$, $-\frac{3}{5}$, $\frac{4}{3}$, etc.; (*b*) $-\frac{4}{5}$, $\frac{3}{5}$, $-\frac{4}{3}$, etc.
3. $-\frac{1}{3}\sqrt{3}$, $-\sqrt{3}$, $\frac{2}{3}\sqrt{3}$, -2
5.

	sin	cos	tan		sin	cos	tan
(*a*)	$\frac{5}{13}$	$\frac{12}{13}$	$\frac{5}{12}$	(*b*)	$\frac{5}{13}$	$-\frac{12}{13}$	$-\frac{5}{12}$
(*c*)	$-\frac{5}{13}$	$-\frac{12}{13}$	$\frac{5}{12}$	(*d*)	$-\frac{5}{13}$	$\frac{12}{13}$	$-\frac{5}{12}$

6.

	sin	cos	tan		sin	cos	tan
(a)	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	(b)	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$
(c)	$-\frac{12}{13}$	$-\frac{5}{13}$	$\frac{12}{5}$	(d)	$-\frac{8}{17}$	$\frac{15}{17}$	$-\frac{8}{15}$
(e)	$-\frac{7}{25}$	$\frac{24}{25}$	$-\frac{7}{24}$	(f)	$\frac{3}{\sqrt{10}}$	$\frac{1}{\sqrt{10}}$	3
(g)	$-\frac{3}{\sqrt{13}}$	$\frac{2}{\sqrt{13}}$	$-\frac{3}{2}$	(h)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	3
(i)	0	-1	∞				

7. (*a*) I, II; (*b*) II, III; (*c*) I, III; (*d*) II, IV; (*e*) II, III; (*f*) I, II
8. (*a*) II; (*b*) IV; (*c*) IV; (*d*) III; (*e*) II; (*f*) IV
10. $-\frac{24}{7}$ 11. 3 12. $-\frac{13}{20}, \frac{37}{20}$

Exercises 4-4, page 86

3. (*a*) 30°, 150°; (*b*) 330°, 210°; (*c*) 30°, 210°; (*d*) 150°, 330°; (*e*) 45°, 315°; (*f*) 135°, 225°
4. (*a*) 90°; (*b*) 180°; (*c*) 0°, 180°; (*d*) 90°, 270°; (*e*) 0°, 180°; (*f*) 270°; (*g*) 90°, 270°; (*h*) 90°, 270°; (*i*) 0°, 180°
6. 2 7. (*a*) $\frac{1}{2}(\sqrt{3}+2)$; (*b*) $\sqrt{2}+\frac{1}{2}$; (*c*) $\frac{5}{2}$; (*d*) $-\frac{5}{2}$
8. 2 16. (*a*) 3; (*b*) 4; (*c*) -2; (*d*) 4

Exercises 4-5, page 90

1. sin 40°, − cos 40°, − tan 40°, etc.
2. − sin 35°, cos 35°, − tan 35°, etc.
3. (*a*) − sin 63°, − cos 63°, tan 63°, etc.
   (*b*) − sin 34°, cos 34°, − tan 34°, etc.
   (*c*) − sin 18°, − cos 18°, tan 18°, etc.
   (*d*) sin 10°, − cos 10°, − tan 10°, etc.
   (*e*) − sin 50°, cos 50°, − tan 50°, etc.
   (*f*) sin 25°, − cos 25°, − tan 25°, etc.
   (*g*) − sin 10°, cos 10°, − tan 10°, etc.
   (*h*) sin 70°, − cos 70°, − tan 70°, etc.
   (*i*) − sin 5°, − cos 5°, tan 5°, etc.
   (*j*) sin 10°, cos 10°, tan 10°, etc.
   (*k*) sin 20°, − cos 20°, − tan 20°, etc.
   (*l*) sin 81°, − cos 81°, − tan 81°, etc.
   (*m*) − sin 80°, − cos 80°, tan 80°, etc.
   (*n*) sin 50°, − cos 50°, − tan 50°, etc.
   (*o*) − sin 25°, − cos 25°, tan 25°, etc.

Exercises 4-6, page 93

1. (*a*) − sin 85°, − cos 85°, tan 85°, etc.
   (*b*) − sin 85°, cos 85°, − tan 85°, etc.
   (*c*) sin 55°, − cos 55°, − tan 55°, etc.
2. cos 5°, − tan 22°, − csc 23°, − cos 17°, − tan 40°, csc 40°, − cos 10°, − cos 20°, cot 30°

3. (*a*) $-\sin\theta$; (*b*) $\cos 2\theta$; (*c*) $-\tan\theta$; (*d*) $-\sec\theta$;
(*e*) $\csc\theta$; (*f*) $-\sin 2\theta$; (*g*) $\cot\theta$; (*h*) $\cos\theta$
4. (*a*) $\sin 20° = \sin 160° = -\sin 200° = -\sin 340° = \cos 70°$
(*e*) $\sec 132° = -\sec 48° = \sec 228° = -\sec 312° = \csc 42°$
6. (*a*) $-\sin^2 25° - \cos^2 86°$; (*b*) 0

Exercises 4-7, page 94

1. $\sin\theta = -\dfrac{2}{\sqrt{13}}$, $\cot\theta = -\frac{3}{2}$, etc.
2. $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, etc.
3. (*a*) 210°, 330°; (*b*) 60°, 240°; (*c*) 135°, 315°; (*d*) 45°, 315°;
(*e*) 210°, 330°; (*f*) 120°, 240°
4. (*a*) $\sin 75°$; (*b*) $-\cos 10°$; (*c*) $\sec 20°$; (*d*) $\cot 62°$;
(*e*) $-\csc 70°$; (*f*) $\tan 4°$
5. (*a*) $\sin 10°$; (*b*) $-\cos 15°$; (*c*) $-\tan 15°$; (*d*) $-\tan 30°$;
(*e*) $-\csc 10°$; (*f*) $-\sec 5°$.
6. (*a*) $-\dfrac{1}{\sqrt{3}}$; (*b*) $-\frac{1}{2}\sqrt{3}$; (*c*) $-\frac{1}{2}\sqrt{3}$; (*d*) $\sqrt{2}$; (*e*) $-\sqrt{2}$; (*f*) $\sqrt{3}$
7. $\frac{1}{4}(1-\sqrt{2})$ **8.** $\dfrac{\sqrt{3}-2}{3}$ **9.** $\sin 80° \cos 80°$
14. -1 **15.** $-\frac{1}{4}(3+2\sqrt{2})$

Exercises 4-8, page 100

3. $\sin(A+B) = \sin C$, $\cos(A+B) = -\cos C$, $\tan(A+B) = -\tan C$
4. (*a*) $-\sin 80°$; (*b*) $-\cot 20°$; (*c*) $-\tan 70°$; (*d*) $-\csc 35°$;
(*e*) $-\cos 40°$; (*f*) $\sin 70°$; (*g*) $-\sec 55°$; (*h*) $\cos 10°25'$;
(*i*) $-\sin 14°28'$; (*j*) $-\cot 13°14'$
5. (*a*) $\cos 20°$; (*b*) $-\tan 80°$; (*c*) $-\sin 60°$;
(*d*) $-\tan 15°$; (*e*) $-\sec 65°$; (*f*) $\cos 60°$
6. (*a*) $\cos\theta$; (*b*) $-\tan\theta$; (*c*) $-\tan\theta$; (*d*) $-\cos\theta$;
(*e*) $\tan\theta$; (*f*) $-\sec\theta$; (*g*) $\sec\theta$; (*h*) $-\sin\theta$
7. (*a*) 0.984, −0.177, −5.539, −0.180; (*b*) −0.582, 0.813, −0.716, −1.397;
(*c*) 0.295, 0.955, 0.309, 3.239
8. (*a*) 3; (*b*) −1; (*c*) $\csc^2\theta$; (*d*) $\cos^2\theta$; (*e*) $-\cot\theta$
9. $-\frac{1}{2}(\sqrt{3}+1)$

Exercises 5-1, page 103

1. (*a*) $\frac{1}{4}\pi$; (*b*) $\frac{1}{3}\pi$; (*c*) $\frac{1}{2}\pi$; (*d*) π; (*e*) $\frac{2}{3}\pi$; (*f*) $\frac{3}{4}\pi$; (*g*) $\frac{1}{8}\pi$; (*h*) $\frac{10}{9}\pi$; (*i*) $\frac{8}{3}\pi$
2. (*a*) 60°; (*b*) 135°; (*c*) 2.5°; (*d*) 210°; (*e*) 1200°; (*f*) 176.40°
3. (*a*) 0.0175; (*b*) 0.0003; (*c*) 0.0611; (*d*) 0.1777; (*e*) 3.1515; (*f*) 5.2423
4. (*a*) 5°44′; (*b*) 143°14′; (*c*) 91°40′; (*d*) 343°46′
5. (*a*) $\frac{1}{3}\sqrt{3}$; (*b*) $\frac{1}{2}\sqrt{3}$; (*c*) $\frac{1}{2}\sqrt{2}$; (*d*) $\sqrt{3}$; (*e*) 1; (*f*) −1; (*g*) $\frac{1}{3}\sqrt{3}$;
(*h*) −2; (*i*) 0

6. (*a*) $\frac{\pi}{6}, \frac{\pi}{72}$; (*b*) $\frac{\pi}{2}, \frac{\pi}{24}$; (*c*) $\frac{3}{2}\pi, \frac{\pi}{8}$; (*d*) $4\pi, \frac{1}{3}\pi$; (*e*) $13\pi, \frac{13\pi}{12}$

7. (*a*) $x = 0, y = 0$; (*b*) $x = 0.3623, y = 1$;
(*c*) $x = 0.1564, y = 0.5858$; (*d*) $x = 3.2982, y = 3.4142$;
(*e*) $x = 4.2360, y = 3.7321$; (*f*) $x = 8.3303, y = 3.7321$;
(*g*) $x = 1.1416, y = 2$; (*h*) $x = 6.2832, y = 4$;
(*i*) $x = 11.4248, y = 2$; (*j*) $x = 12.5664, y = 0$;
(*k*) $x = 43.9823, y = 4$

8. (*a*) $x = 5, y = 0$; (*b*) $x = 7.0345, y = 1.7122$;
(*c*) $x = -13.4930, y = 13.3610$

9. 91°21′ **10.** (*a*) $\frac{1}{6}(4\sqrt{3} - 27)$; (*b*) 0; (*c*) 0

11. $-\cos^2 x - \sin^2 x \tan x$

Exercises 5-2, page 105

1. (*a*) 226.20 ft.; (*b*) 358.14 ft.; (*c*) 217.92 ft.;
(*d*) 8.48 ft.; (*e*) 4.2935 ft.; (*f*) $4a$ ft.

2. (*a*) 36°; (*b*) 1°12′; (*c*) 7′12″; (*d*) 1°26′; (*e*) 336°50′

4. 7.5 ft. **5.** 94°4′ **6.** 75 yd. **7.** $\frac{1}{33}$

8. 247.6 revolutions per minute, 25.882 radians per second

9. 69.09 miles, 932.7 miles **10.** 2160 miles **11.** 2.227 miles

12. 62.86 radians per second **13.** 1760 radians per minute

14. 17.05 miles per hour **15.** 7.33 ft. per second

16. 846.4 ft. **17.** 222.7 ft., 4584 ft. **18.** 589.3 ft.

19. 20.94 ft., 200 ft. **20.** 2.963 mils

Exercises 5-3, page 109

1. 20 mils **2.** 3.75 mils **3.** 3000 m. **4.** 80 mils, 60 mils

5. 0.0009818, 1018.1 **6.** 72 yd. **7.** 0.01571

9. 2500 m. **10.** 2500 yd. **11.** 4275 yd. **14.** 294.5 ft.

Exercises 5-4, page 120

1. (*a*) $\frac{2}{5}\pi$; (*b*) $\frac{1}{4}\pi$; (*c*) 2π; (*d*) $\frac{1}{4}\pi$; (*e*) $\frac{1}{3}\pi$; (*f*) π; (*g*) $\frac{\pi}{2}$;
(*h*) 1; (*i*) 3π; (*j*) $\frac{2}{3}\pi$; (*k*) $\frac{2}{3}\pi$; (*l*) π; (*m*) π; (*n*) $\frac{2\pi}{277}$

2. (*a*) 1; (*b*) 4; (*c*) $\frac{1}{2}$; (*d*) 8.6; (*e*) 334; (*f*) $\frac{3}{16}$; (*g*) 1; (*h*) 8

10. $\frac{2\pi}{377}$, 110

Exercises 5-5, page 121

1. $\frac{\pi}{18}, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, -\frac{3}{2}\pi, -\frac{\pi}{10}, -0.4232$

4. 60°, 180°, 120°, 315°, 114°36′, 286°29′, −171°53′

5. (*a*) − tan 30°; (*b*) cos 25°43′; (*c*) − cot 36°; (*d*) − csc 25°43′

6. (*a*) 2.4; (*b*) 137°30′ **7.** 3.35 ft.

8. 0.42 radian **9.** 18.40 miles per second
10. 30.16 radians per second, 754.0 ft. per min.
11. (*a*) 0; (*b*) 2; (*c*) 1; (*d*) 0; (*e*) -3.9793; (*f*) $-\sqrt{3}$; (*g*) 8
12. (*a*) $\cos^2 x - \sin^2 x$; (*b*) 1; (*c*) $\cot^2 A$; (*d*) 1; (*e*) $-\cos^2 \theta$; (*f*) 0; (*g*) 1
18. $\dfrac{2c}{(c^2 - 1)\sqrt{c^2 + 1}}$
19. $\sin(-\theta) = \frac{15}{17}$, $\cos(-6) = -\frac{8}{17}$, $\tan(-\theta) = -\frac{15}{8}$, etc.
20. $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\cos\theta = -\dfrac{2}{\sqrt{5}}$, $\tan\theta = -\frac{1}{2}$, etc.
21. $\frac{119}{169}$ **22.** $-\frac{2}{5}$ **27.** $(n - 2)\pi$
28. 523.6 ft. per sec. **29.** 92,800,000 miles
30. 830.8 ft. **31.** 182.4 ft. **32.** 304.1 ft.

Exercises 6-1, page 127

1. $\frac{2}{9}(1 + \sqrt{10})$, $\frac{1}{9}(4\sqrt{2} - \sqrt{5})$
3. $\frac{1}{4}\sqrt{2}(\sqrt{3} + 1)$, $\frac{1}{4}\sqrt{2}(\sqrt{3} - 1)$, etc.
4. $\frac{1}{4}\sqrt{2}(\sqrt{3} + 1)$, etc. **5.** 0 **6.** (*b*) 0.0178
7. $\frac{33}{65}$ **8.** $\frac{4}{5}$, $\frac{3}{5}$
11. (*a*) $\cos y$, $-\sin y$; (*b*) $\sin y$, $-\cos y$; (*c*) $-\sin y$, $-\cos y$;
(*d*) $-\cos y$, $-\sin y$; (*e*) $-\cos y$, $\sin y$; (*f*) $-\sin y$, $\cos y$;
(*g*) $\sin y$, $\cos y$; (*h*) $-\cos x$, $\sin x$; (*i*) $-\sin x$, $-\cos x$;
(*j*) $\cos x$, $-\sin x$; (*k*) $-\sin y$, $\cos y$;
(*l*) $\dfrac{1}{\sqrt{2}}(\cos y - \sin y)$, $\dfrac{1}{\sqrt{2}}(\cos y + \sin y)$;
(*m*) $-\dfrac{1}{\sqrt{2}}(\cos y + \sin y)$, $-\dfrac{1}{\sqrt{2}}(\cos y - \sin y)$;
(*n*) $\frac{1}{2}(\cos y + \sqrt{3}\sin y)$, $\frac{1}{2}(\sqrt{3}\cos y - \sin y)$;
(*o*) $\frac{1}{2}(\sqrt{3}\cos y - \sin y)$, $\frac{1}{2}(\cos y + \sqrt{3}\sin y)$
15. $\dfrac{1}{2\sqrt{3}}(\sqrt{3} + 2)$ **24.** $3\sin\theta - 4\sin^3\theta$ **25.** $4\cos^3\theta - 3\cos\theta$

Exercises 6-2, page 131

3. $-(2 + \sqrt{3})$
5. $\sin(\alpha + \beta) = -\frac{33}{65}$, $\cos(\alpha + \beta) = \frac{56}{65}$, $\tan(\alpha + \beta) = -\frac{33}{56}$, etc.
6. $\sin(\alpha - \beta) = -\frac{308}{533}$, $\cos(\alpha - \beta) = -\frac{435}{533}$, $\tan(\alpha - \beta) = \frac{308}{435}$, etc.
7. $-\frac{1}{2}$ **8.** 3
14. (*a*) $\sin 5x$; (*b*) $\cos x$; (*c*) $\sin x$; (*d*) 0; (*e*) $\cos 2x$; (*f*) $\sin 2x$
15. (*a*) $\tan 5x$; (*b*) $\tan 2x$
20. (*a*) $4\sin(\theta + 30°)$; (*b*) $\sqrt{2}a\sin(\theta + 45°)$;
(*c*) $\sin(\theta + 45°)$; (*d*) $2\sqrt{3}\sin(\theta - 30°)$;
(*e*) $5\sin(\theta + 53°8')$; (*f*) $2\cos(\theta + 45°)$

Exercises 6-3, page 135

1. $-\frac{24}{25}, \frac{7}{25}, -\frac{24}{7}, \frac{3}{10}\sqrt{10}, \frac{1}{10}\sqrt{10}, 3$
2. $\frac{1}{2}\sqrt{2-\sqrt{2}}, \frac{1}{2}\sqrt{2+\sqrt{2}}$
6. $\pm(4\sin x - 8\sin^3 x)\sqrt{1-\sin^2 x}, \dfrac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$
8. $\frac{1}{4}(\sqrt{5}-1)$ **9.** $-\frac{119}{120}, \frac{5}{13}, \frac{120}{169}, -\frac{169}{120}$

Exercises 6-4, page 139

1. (*a*) $2\sin 30^\circ \cos 5^\circ$; (*b*) $2\cos 37^\circ 30' \sin 7^\circ 30'$;
(*c*) $2\cos 45^\circ \cos 20^\circ$; (*d*) $-2\sin 40^\circ \sin 35^\circ$;
(*e*) $2\cos 3x \cos x$; (*f*) $2\cos\dfrac{7x}{2}\sin\dfrac{3x}{2}$;
(*g*) $2\sin 2x \cos x$; (*h*) $-2\sin 4x \sin x$
2. (*a*) $\frac{1}{2}(\sin 10x - \sin 4x)$; (*b*) $\frac{1}{2}(\cos 10x + \cos 4x)$;
(*c*) $\frac{1}{4}(\cos 2x + \cos 4x - \cos 6x - 1)$;
(*d*) $\frac{1}{4}(\sin 15x + \sin 9x + \sin 5x - \sin x)$

Exercises 6-5, page 141

2. (*a*) $\frac{56}{65}$; (*b*) $-\frac{63}{16}$; (*c*) $\frac{33}{65}$; (*d*) $-\frac{16}{63}$; (*e*) $-\frac{56}{33}$; (*f*) $\frac{33}{56}$
5. $\frac{6}{7}, \frac{2}{9}$ **39.** Varies from 0 to 1

Exercises 7-1, page 147

1. (*a*) $x = y = 3\sqrt{3}$; (*b*) $x = 18, y = 18\sqrt{3}$
2. (*a*) $x = 35\sin 60^\circ \csc 70^\circ, y = 35\sin 50^\circ \csc 70^\circ$;
(*b*) $x = y = 35\sin 70^\circ \csc 40^\circ$; (*c*) $x = 40\sin 111^\circ 20' \csc 30'$;
(*d*) $x = 60\sin 74^\circ 25' \csc 40^\circ, y = 60\sin 25^\circ 35' \csc 40'$
3. $x = \csc 30^\circ \sin 80^\circ, y = \csc 30^\circ \sin 50^\circ$,
$z = \csc 30^\circ \sin 50^\circ \sin 80^\circ \csc 60^\circ, p = \csc 30^\circ \sin 50^\circ \sin 40^\circ \csc 60^\circ$
4. $\sin B = 0.6862, x = 624\sin(118^\circ - B)\csc 62^\circ$
5. $[312\sin(118^\circ - B)(\csc 62^\circ)]\ 485\sin 62^\circ$
6. (*a*) $x = a\sin 65^\circ \csc 40^\circ, y = a\sin 75^\circ \csc 40^\circ$;
(*b*) $x = a\csc\theta\sin(\theta+\varphi), y = a\csc\theta\sin\varphi$
7. $x = \sin 50^\circ \csc 60^\circ, z = \sin 50^\circ \csc 30^\circ, w = \sin 50^\circ \csc 70^\circ$,
$y = \sin^2 50^\circ \csc 60^\circ \csc 70^\circ$

Exercises 7-2, page 151

1. $b = 4.422$	**2.** $b = 4383$	**3.** $a = 895.4$
$c = 1.730$	$c = 6135$	$b = 728.5$
$C = 22^\circ 24'$	$A = 81^\circ 47'$	$C = 67^\circ 35'$
4. $a = 177.5$	**5.** $a = 241.0$	**6.** $b = 695.0$
$b = 213.7$	$b = 165.5$	$c = 345.4$
$B = 62^\circ 24'$	$C = 68^\circ 15'$	$C = 21^\circ 14'$
7. 345.4 ft.	**8.** 73.55 ft.	**10.** (*a*) 3.113; (*b*) 51,767

11. 26,624 ft., 26,689 ft. **12.** 2232.2 ft.
13. 590.43 ft. **14.** 192.4 ft.

Exercises 7-3, page 156

1. $B_1 = 24°57'$, $B_2 = 155°3'$; $C_1 = 133°49'$, $C_2 = 3°43'$; $c_1 = 615.7$, $c_2 = 55.31$
2. $A_1 = 134°17'$, $A_2 = 3°9'$; $C_1 = 24°26'$, $C_2 = 155°34'$; $a_1 = 623.3$, $a_2 = 47.84$
3. $B_1 = 51°9'$, $B_2 = 128°51'$; $C_1 = 87°38'$, $C_2 = 9°56'$; $c_1 = 116.8$, $c_2 = 20.17$
4. $a_1 = 167.5$, $a_2 = 35.2$; $A_1 = 81°36'$, $A_2 = 12°0'$; $C_1 = 55°12'$, $C_2 = 124°48'$
5. $B = 36°27'$; $C = 76°1'$; $c = 308.6$
6. $a = 31.67$, $C = 90°$; $A = 23°48'$
7. $B = 26°13'$; $C = 117°23'$; $c = 72.1$
8. $c_1 = 60.3$, $c_2 = 24.56$; $B_1 = 56°21'$, $B_2 = 123°39'$; $C_1 = 91°20'$, $C_2 = 24°2'$
9. $B_1 = 79°7'$, $B_2 = 100°53'$; $C_1 = 46°35'$, $C_2 = 24°19'$; $c_1 = 3.79$, $c_2 = 2.148$
10. $A_1 = 99°1'$, $A_2 = 9°47'$; $B_1 = 45°23'$, $B_2 = 134°37'$; $a_1 = 300.5$, $a_2 = 51.69$
11. No solution
12. No solution
13. 17,091
14. 47°48′
15. 1458 ft.

Exercises 7-4, page 161

1. $A = 77°13'$; $B = 43°30'$; $c = 14.99$
2. $A = 86°23'$; $B = 30°1'$; $c = 671.4$
3. $B = 67°38'$; $C = 51°10'$; $a = 220.1$
4. $A = 40°28'$; $B = 99°52'$; $c = 27.46$
5. $B = 51°56'$; $C = 77°24'$; $a = 83.7$
6. $A = 92°52'$; $B = 22°30'$; $c = 0.5365$
7. $A = 52°10'$; $B = 17°18'$; $c = 7.398$
8. $A = 46°49.8'$; $B = 22°29.2'$; $c = 45.21$
9. 39.25 ft.
10. 5120 ft.
11. 147.97 ft.
12. 4064 ft., 165°53′
14. 7.22 nautical miles, 197°1′
15. 5281 ft.
16. 9.16 miles, 29.89 miles
17. 443.2 ft.

Exercises 7-5, page 163

2. (a) 7; (b) 18.51; (c) 9.54; (d) 3.14; (e) 184.5; (f) 5.2; (g) 11.7
3. (a) 87°13.5′, 53°2.5′, 39°44′; (b) 44°25′, 57°7′, 78°28′; (c) 104°29′, 28°57′, 46°34′; (d) 82°49′, 55°46′, 41°25′
5. 7.18, 16.3

Exercises 7-6, page 168

1. $A = 106°47'$; $B = 46°53'$; $C = 26°20'$
2. $A = 27°48'$; $B = 33°46'$; $C = 118°28'$
3. $A = 27°20'$; $B = 143°8'$; $C = 9°32'$
4. $A = 8°20'$; $B = 33°42'$; $C = 137°58'$
5. $A = 44°42'$; $B = 49°36'$; $C = 85°42'$
6. $A = 51°54'$; $B = 59°30'$; $C = 68°36'$
7. $A = 28°8'$; $B = 114°58'$; $C = 36°54'$
8. $A = 45°38'$; $B = 75°20'$; $C = 59°2'$
9. $A = 80.4°$; $B = 56.6°$; $C = 43.0°$

10. 496 ft. **11.** 7.682 miles, 9.006 miles
13. (*a*) 1.967; (*b*) 1.288 **14.** 2551 sq. ft.

Exercises 7-7, page 170

1. (*a*) 75; (*b*) 109; (*c*) 28.2; (*d*) 16.1; (*e*) 85.4; (*f*) 239.4; (*g*) 654.5; (*h*) 17.15; (*i*) 156.3; (*j*) 68.05
2. 189.9 **3.** 23.97 **4.** 165.2 **5.** 40154.5 sq. ft. **6.** 408.8 ft.

Exercises 7-8, page 170

1. $\sqrt{52}, \dfrac{6 \sin 60°}{x}, \dfrac{8 \sin 60°}{x}$ **2.** $\dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$

3. $\sqrt{34 - 15\sqrt{3}}, \sin A = \dfrac{3 \sin 30°}{x}, \sin B = \dfrac{5 \sin 30°}{x}$

$\sqrt{52 - 24\sqrt{2}}, \sin A = \dfrac{6 \sin 45°}{x}, \sin B = \dfrac{4 \sin 45°}{x}$

4. $\frac{1}{7} \tan 45°, 0$ **5.** $\sqrt{1873 - 924\sqrt{2}}$ **6.** $\frac{5}{61} \tan 67°30'$

7. 326.7 **9.** $\dfrac{c^2 \sin A \sin B}{2 \sin (A + B)}$ **10.** $\frac{9}{16}$

11. (*a*) $8 \sin 60° \sin 40° \csc 50° \csc 35°$; (*b*) 10.14
12. $h = m \sin w \csc (w + z) \sin y \csc (x + y)$

13. $x = \dfrac{\sin 40°}{\sin 75°}, y = \dfrac{\sin 35°}{\sin 75°}, q = 1, p = 1, w = \dfrac{\sin 80°}{\sin 40°} - \dfrac{\sin 35°}{\sin 75°},$

$z = \dfrac{\sin 70°}{\sin 35°} - \dfrac{\sin 40°}{\sin 75°}$

17. $AC = \dfrac{\sin \beta}{\sin \varphi}, OC = \dfrac{\sin (\beta + \varphi)}{\sin \varphi}, OB = \dfrac{\sin \theta}{\sin (\gamma + \theta)}, AB = \dfrac{\sin \gamma}{\sin (\gamma + \theta)}$

18. 1474 ft., 1253 ft. **19.** 6330 ft. **20.** 84°8′
21. 722.2 ft. **22.** 65.3 miles, 26°49′ **24.** 52.4
24. 3.2 min., 2.2 nautical miles **26.** 373 ft.
27. 3.2 miles per hour **28.** 731 ft., 50°38′ **29.** 6461 ft.
30. 88 ft. **31.** 219.8 ft. **32.** 8 nautical miles
33. 4°44′ **34.** 231.7 ft., 328.7 ft. **40.** 3162
41. 2554 ft. **42.** 52.84 ft. **43.** 2109 yd. **44.** 9.72 knots
45. 85.6 ft. **46.** 509 yd. **48.** 107 ft. **51.** 7 lb., 21°48′
52. 189.8 nautical miles, 293° **53.** (*a*) 16.7 lb.; (*b*) 9.02 lb.
56. PB = 403 yd., PA = 140 yd., PC = 734 yd. **57.** 79.4 yd., 1°49′
59. 269°55′, 75.2 miles **60.** 52.1 miles, 333°39′ **61.** 359°54′, 19.3 miles

Exercises 8-1, page 181

1. 30°, 150° **2.** 60°, 120° **3.** 225°, 315°
4. 60°, 240° **5** 135°, 315° **6.** 120°, 240°

7. $135°$, $225°$ 8. $45°$, $315°$ 9. $60°$, $300°$
10. $210°$, $330°$ 11. $60°$, $120°$ 12. $25°36'$, $154°24'$

Exercises 8-3, page 183

1. (a) $\frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$; (b) $\frac{\pi}{3} + 2n\pi, \frac{2\pi}{3} + 2n\pi$; (c) $\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi$;
(d) $\frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$; (e) $2n\pi$, $\pi + 2n\pi$; (f) $\frac{3\pi}{2} + 2n\pi$;
(g) $19°28' + n360°$, $160°32' + n360°$;
(h) $25°36' + n360°$, $154°24' + n360°$;
(i) $204°37' + n360°$, $335°23' + n360°$;
(j) $\frac{\pi}{4} + 2n\pi, \frac{7\pi}{4} + 2n\pi$; (k) $\frac{3\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi$;
(l) $\frac{5\pi}{6} + 2n\pi, \frac{7\pi}{6} + 2n\pi$; (m) $\frac{7\pi}{6} + 2n\pi, \frac{11\pi}{6} + 2n\pi$;
(n) $\frac{3\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi$; (o) $\frac{\pi}{2} + n\pi$;
(p) $\frac{\pi}{4} + 2n\pi, \frac{5\pi}{4} + 2n\pi$; (q) $n\pi$;
(r) $66°38' + n360°$, $246°38' + n360°$
2. (a) $\frac{11\pi}{6} + 2n\pi$; (b) $\frac{7\pi}{6} + 2n\pi$; (c) $\frac{3\pi}{4} + 2n\pi$; (d) $\frac{5\pi}{4} + 2n\pi$;
(e) $\frac{5\pi}{6} + 2n\pi$; (f) $\frac{5\pi}{3} + 2n\pi$
3. (a) $21°6' + n360$, $158°54' + n360°$;
(b) $53°8' + n360°$, $306°52' + n360°$;
(c) $41°59' + n360°$, $221°59' + n360°$;
(d) $25°28' + n360°$, $205°28' + n360°$;
(e) $73°0' + n360°$, $287°0' + n360°$;
(f) $55°44' + n360°$, $124°16' + n360°$;
(g) $53°8' + n360°$, $306°52' + n360°$;
(h) $41°49' + n360°$, $138°11' + n360°$;
(i) $51°20' + n360°$, $231°20' + n360°$;
(j) $48°11' + n360°$, $311°49' + n360°$;
(k) $48°49' + n360°$, $228°49' + n360°$;
(l) $3°49' + n360°$, $176°11' + n360°$

Exercises 8-4, page 185

1. (a) $\frac{\pi}{4}$; (b) $\frac{\pi}{3}$; (c) 0; (d) $\frac{\pi}{4}$; (e) $\frac{\pi}{3}$; (f) 0; (g) $\frac{\pi}{4}$; (h) $\frac{\pi}{3}$; (i) $\frac{\pi}{4}$; (j) $\frac{\pi}{2}$;
(k) $\frac{\pi}{6}$; (l) $\frac{\pi}{3}$; (m) $\frac{\pi}{2}$; (n) $\frac{\pi}{6}$; (o) $\frac{\pi}{3}$; (p) 0; (q) $\frac{\pi}{6}$; (r) $\frac{\pi}{3}$
2. (a) $-30°$; (b) $-45°$; (c) $-60°$; (d) $-45°$; (e) $-60°$; (f) $-30°$
3. (a) $135°$; (b) $150°$; (c) $120°$; (d) $135°$; (e) $150°$; (f) $120°$

4. (*a*) -30; (*b*) $45°$; (*c*) $-60°$; (*d*) $90°$; (*e*) $-135°$; (*f*) $-180°$; (*g*) $-45°$; (*h*) $60°$; (*i*) $60°$; (*j*) $-135°$; (*k*) $180°$

5. (*a*) $\frac{\pi}{3}$; (*b*) $-\frac{\pi}{6}$; (*c*) $\frac{\pi}{6}$; (*d*) $-\frac{\pi}{3}$; (*e*) π; (*f*) $-\frac{2\pi}{3}$

Exercises 8-5, page 189

1. $\frac{2}{3}$ **2.** $\frac{3}{5}$ **3.** $\frac{1}{12}\sqrt{119}$ **4.** $\frac{1}{3}\sqrt{5}$ **5.** $-\sqrt{\frac{8}{7}}$

6. $-\frac{4}{5}$ **7.** $-\frac{3}{5}$ **8.** $-\frac{3}{5}$ **9.** $\frac{2}{\sqrt{5}}$ **10.** $\frac{1}{2}\sqrt{5}$

11. ± 1 **12.** $\frac{\sqrt{30.16}}{5.4}$ **13.** 0 **14.** $\frac{4}{3}$ **15.** $\frac{4}{\sqrt{17}}$

16. (*a*) $-\frac{1}{8}$; (*b*) $\frac{2}{\sqrt{3}}$; (*c*) 1; (*d*) -0.993

Exercises 8-6, page 191

1. (*a*) $30°, 150°, 210°, 330°$; (*b*) $45°, 135°, 225°, 315°$;
(*c*) $60°, 120°, 240°, 300°$; (*d*) $60°, 120°, 240°, 300°$;
(*e*) $22\frac{1}{2}°, 112\frac{1}{2}°, 202\frac{1}{2}°, 292\frac{1}{2}°$; (*f*) $10°, 50°, 130°, 170°, 250°, 290°$

2. (*a*) $120°, 240°$; (*b*) $30°, 150°, 210°, 330°$;
(*c*) $60°, 120°$; (*d*) $60°, 300°$;
(*e*) $30°, 150°, 210°, 330°$; (*f*) $45°, 225°$ (*g*) $135°, 315°$

3. (*a*) $\frac{1}{3}\pi, \frac{3}{4}\pi, \frac{4}{3}\pi, \frac{7}{4}\pi$; (*b*) $\frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi$;
(*c*) $\frac{1}{6}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$; (*d*) $\frac{1}{2}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi, \frac{3}{2}\pi$

4. (*a*) $n360°, 120° + n360°, 240° + n360°$;
(*b*) $30° + n360°, 150° + n360°$; (*c*) $270° + n360°$;
(*d*) $45° + n180°, 105° + n180°, 165° + n180°$;
(*e*) $56°19' + n180°, 135° + n180°$; (*f*) $33°41' + n180°, 45° + n180°$;
(*g*) $37°59' + n45°$; (*h*) $90° + n180°, \pm 60° + n180°, \pm 120° + n180°$;
(*i*) $51°19' + n360°, 308°41' + n360°, 180° + n360°$;
(*j*) $30° + n360°, 150° + n360°, 90° + n360°$;
(*k*) $45° + n90°$; (*l*) $45° + n180°, 71°34' + n180°$;
(*m*) $120° + n360°, 240° + n360°$; (*n*) $9°44' + n360°, 151° + n360°$;
(*o*) $n360°, 90° + n360°$;(*p*) $60° + n360°$;
(*q*) $105° + n180°, 165° + n180°$;
(*r*) $90° + n180°, 120° + n360°, 240° + n360°$;
(*s*) $30° + n180°, 150° + n180°$

5. (*a*) $n180°, \pm 60° + n360°$; (*b*) $90° + n180°, 30° + n360°, 150° + n360°$;
(*c*) $n180°, \pm 60° + n180°, \pm 120° + n180°$;
(*d*) $90° + n180°, 210° + n360°, 330° + n360°$;
(*e*) $45° + n90°, 15° + n180°, 75° + n180°$;
(*f*) $30° + n360°, 330° + n360°, n180°$;
(*g*) $n90°, 30° + n90°; 60° + n90°$;
(*h*) $n90°, 52°14' + n180°, 127°46' + n180°$; (*i*) $n180°, \pm 60° + n180°$

6. (*a*) $n\pi$; (*b*) $2n\pi, \frac{2}{3}\pi + 2n\pi, \frac{4}{3}\pi + 2n\pi$; (*c*) $n\pi$; (*d*) $n\pi$

Exercises 8-7, page 193

1. (*a*) $\pm\frac{5}{13}$; (*b*) $\pm\dfrac{1}{\sqrt{2}}$; (*c*) $\dfrac{2a}{1-a^2}$; (*d*) $\frac{7}{24}$;

(*e*) $2a^2-1$; (*f*) $\dfrac{1}{\sqrt{a^2+1}}$; (*g*) $n\pi+\dfrac{\pi}{6}$; (*h*) $n\pi\pm\dfrac{\pi}{4}$

3. (*a*) $71°34' + n360°$, $251°34' + n360°$;
(*b*) $158°32' + n360°$, $201°28' + n360°$; (*c*) $n180°$

4. (*a*) $199°28' + n360°$, $340° + n360°$;
(*b*) $70°32' + n360°$, $289°28' + n360°$; (*c*) $45° + n180°$, $116°34' + n180°$;
(*d*) $210° + n360°$, $330° + n360°$, $41°49' + n360°$, $138°11' + n360°$;
(*e*) $90° + n180°$, $210° + n360°$, $330° + n360°$;
(*f*) $204°28' + n360°$, $335°32' + n360°$;
(*g*) $76°49' + n180°$, $347°3' + n180°$; (*h*) $135° + n180°$;
(*i*) $270° + n360°$, $126°52' + n360°$; (*j*) $n360°$;
(*k*) $60° + n360°$; (*l*) $30° + n90°$, $35°16' + n90°$

5. (*a*) $n360°$, $106°16' + n360°$; (*b*) $77°20' + n360°$, $180° + n360°$

6. (*a*) $240° + n360°$, $300° + n360°$; (*b*) $210° + n360°$, $330° + n360°$;
(*c*) $\pm 30° - n180°$; (*d*) $49°21' + n360°$, $310°29' + n360°$;
(*e*) $\pm 60° + n720°$, $\pm 300° + n720°$

7. (*d*) $\dfrac{(x-a)^2}{b^2}+\dfrac{(y-c)^2}{d^2}=1$; (*e*) $\left(\dfrac{y}{b}\right)^{\frac{2}{3}}-\left(\dfrac{x}{a}\right)^{\frac{2}{3}}=1$

8. (*a*) $\frac{1}{2}$; (*b*) $\sqrt{3}$; (*c*) $\dfrac{\sqrt{10}}{2}$; (*d*) $\sqrt{3}$;

(*e*) none; (*f*) $\frac{1}{4}$; (*g*) $\dfrac{\sqrt{21}}{4}$; (*h*) 13

Exercises 9-1, page 198

3. Each side $= 5\pi$ in. **5.** 3000 miles, 3638 miles, 2750.3 miles

8. (*a*) $c = 30°$, $a = 90°$, $b = 90°$

Exercises 9-2, page 203

1. (*a*) $c = \cos^{-1}\dfrac{\sqrt{3}}{4}$; (*b*) $B = \sec^{-1}\sqrt{3}$;
(*c*) $c = \tan^{-1} 2$; (*d*) $A = \sec^{-1} 4$;
(*e*) $b = \tan^{-1}\sqrt{\frac{3}{2}}$; (*f*) impossible

3. (*a*) $A = \tan^{-1} 2$; (*b*) impossible;
(*c*) $a = \tan^{-1}\frac{3}{2}$; (*d*) $c = \pi - \sec^{-1}\sqrt{3}$;
(*e*) $A = \cos^{-1}\frac{3}{4}$; (*f*) $B = \sec^{-1}\sqrt{3}$

Exercises 9-4, page 208

1. $b = 2°14'$, $c = 10°46'$, $A = 78°9'$

2. $a = 44°44'$, $b = 14°59'$, $A = 75°22'$

3. $b = 10°49'$, $c = 118°20'$, $A = 95°55'$

4. $A = 52°16'$, $B = 57°26'$, $b = 47°7'$

5. $a = 58°21'$, $A = 65°11'$, $B = 53°7'$

6. $A = 155°46'$, $B = 68°41'$, $b = 27°38'$
7. $a = 127°4'$, $b = 49°59'$, $A = 120°3'$
8. $a = 22°16'$, $b = 24°24'$, $B = 50°8'$
9. $a = 119°59'$, $b = 120°10'$, $C = 75°28'$
10. $a = 50°0'$, $b = 56°51'$, $B = 63°25'$
11. $b = 51°52'$, $A = 27°29'$, $B = 73°27'$
12. $c = 54°19'$, $A = 47°0'$, $B = 57°59'$
13. $A = 54°1'$, $b = 155°28'$, $c = 142°9'$
14. $c = 133°33'$, $A = 126°40'$, $B = 47°12'$
15. $c = 54°19'$, $A = 57°59'$, $B = 47°0'$
16. $a = 50°1'$, $b = 143°4'$, $c = 120°55'$
17. $a = 67°33'$, $b = 100°45'$, $c = 94°5'$
18. $a = 51°52'$, $A = 73°27'$, $B = 27°29'$
19. $A = 118°21'$, $b = 96°22'$, $c = 86°58'$
20. $a = 50°0'$, $c = 91°47'$, $B = 92°8'$
22. $D = 691$ miles, $L_2 = 39°31'$, $C = 80°19'$
24. $B = 53°48'$

Exercises 9-5, page 210

1. $a_1 = 69°50'$, $c_1 = 73°45'$, $A_1 = 77°53'$
   $a_2 = 110°10'$, $c_2 = 106°15'$, $A_2 = 102°7'$
2. $b_1 = 28°16'$, $c_1 = 78°50'$, $B_1 = 28°51'$
   $b_2 = 121°44'$, $c_2 = 101°10'$, $B_2 = 151°9'$
3. $a_1 = 18°55'$, $c_1 = 127°1'$, $A_1 = 23°55'$
   $a_2 = 161°5'$, $c_2 = 52°59'$, $A_2 = 156°5'$
4. $b_1 = 39°5'$, $c_1 = 136°50'$, $B_1 = 68°0'$
   $b_2 = 140°55'$, $c_2 = 43°10'$, $B_2 = 112°0'$
5. $a_1 = 25°59'$, $c_1 = 33°20'$, $A_1 = 52°52'$
   $a_2 = 154°1'$, $c_2 = 146°40'$, $A_2 = 127°8'$
6. $a_1 = 60°34'$, $c_1 = 68°40'$, $A_1 = 69°14'$
   $a_2 = 119°26'$, $c_2 = 111°20'$, $A_2 = 110°46'$

Exercises 9-6, page 211

3. (*a*) $a = 29°43'$, $c = 143°52'$, $B = 141°23'$;
   (*b*) $a' = 133°10'$, $B' = 108°18'$, $c' = 73°35'$

Exercises 9-7, page 213

1. $a = 68°36'$, $b = 59°19'$, $C = 103°26'$
2. $a = 67°47'$, $b = 78°21'$, $B = 77°24'$
3. $b = 117°45'$, $A = 96°27'$, $C = 93°1'$
4. $a = 94°23'$, $b = 69°49'$, $C = 88°23'$
5. $a = 106°57'$, $B = 8°50'$, $C = 28°3'$
6. $A = 105°20'$, $B = 160°14'$, $C = 104°24'$

Exercises 9-8, page 215

1. $c = 120°11'$
   $A = 65°12'$
   $B = 49°28'$

2. $b = 100°48'$
   $A = 96°0'$
   $C = 125°43'$

3. $a = 69°34'$
$B = 135°5'$
$C = 50°30'$

4. $c = 108°39'$
$B = 40°23'$
$A = 64°49'$

5. $c = 104°13'$
$B = 51°47'$
$A = 63°48'$

6. $a = 65°29'$
$B = 148°15'$
$C = 44°9'$

7. $a = 145°25'$
$b = 139°46'$
$C = 49°46'$

8. $a = 23°57'$
$c = 118°2'$
$B = 102°6'$

9. $B_1 = 42°38'$
$B_2 = 137°22'$
$C_1 = 160°2'$
$C_2 = 50°19'$
$c_1 = 153°39'$
$c_2 = 90°5'$

10. $B_1 = 120°47'$
$B_2 = 59°12'$
$C_1 = 97°43'$
$C_2 = 29°9'$
$c_1 = 55°42'$
$c_2 = 23°57'$

11. $B = 131°25'$
$C = 108°19'$
$c = 78°21'$

12. $C_1 = 59°24'$
$C_2 = 120°36'$
$B_1 = 115°40'$
$B_2 = 27°0'$
$b_1 = 97°33'$
$b_2 = 29°58'$

13. (*a*) $b = 76°47'$
$a = 96°46'$
$A = 99°42'$

(*b*) $b_1 = 109°50'$
$b_2 = 70°10'$
$c_1 = 98°21'$
$c_2 = 168°49'$
$C_1 = 109°55'$
$C_2 = 169°23'$

Exercises 10-2, page 224

1. 127.2 miles, 141.2 miles
2. 65.71 miles
3. 23.34 miles, 161.1 miles
4. $2°35'$
5. 101.3 miles
6. $L = 37°26.8'$, $\lambda = 56°22.4'$
7. 231.2°, 201.1 miles
8. 316°, 239 miles
10. $8°56'$, $8°57'$

Exercises 10-3, page 227

9. $L = 30°35'$, $\lambda = 38°31'$
10. (*a*) $241°50'$; (*b*) $259°16'$; (*c*) $38°50'$; (*d*) $224°20'$
11. (*a*) 1800 miles; (*b*) 2990 miles; (*c*) 1620 miles

Exercises 11-1, page 237

3. (*a*) $A = 71°23'$; (*b*) $B = 53°37'$
4. (*a*) $b = 44°14'$; (*b*) $B = 131°18'$

Exercises 11-2, page 240

1. (*a*) $a = 42°20'$; (*b*) $a = 64°11'$; (*c*) $a = 100°11'$
2. (*a*) $a = 78°40'$; (*b*) $b = 68°18'$; (*c*) $c = 108°10'$;
(*d*) $b = 35°6'$; (*e*) $c = 80°14'$; (*f*) $b = 121°48'$;
(*g*) $A = 58°24'$; (*h*) $B = 70°31'$; (*i*) $C = 107°26'$;
(*j*) $C = 72°46'$; (*k*) $A = 55°31'$; (*l*) $B = 120°7'$

Exercises 11-3, page 242

1. (a) $A = 137°40'$; (b) $A = 79°49'$; (c) $A = 117°18'$
2. (a) $A = 71°7'$; (b) $B = 112°$; (c) $C = 94°38'$; (d) $a = 49°2'$; (e) $b = 61°1'$; (f) $c = 81°27'$

Exercises 11-4, page 246

1. (a) $A = 33°11'$, $B = 50°44'$, $C = 108°32'$
   (b) $A = 34°47'$, $B = 81°6'$, $C = 81°6'$
   (c) $A = 145°13'$, $B = 98°54'$, $C = 81°6'$
   (d) $A = 118°44'$, $B = 29°38'$, $C = 68°8'$
   (e) $A = 123°54'$, $B = 57°47'$, $C = 46°52'$
   (f) $A = 81°52'$, $B = 97°32'$, $C = 111°2'$
2. (a) $a = 76°10'$, $b = 127°33'$, $c = 76°10'$
   (b) $a = 146°49'$, $b = 71°28'$, $c = 129°16'$
   (c) $a = 56°52'$, $b = 126°58'$, $c = 139°21'$
   (d) $a = 51°18'$, $b = 64°3'$, $c = 51°18'$
   (e) $a = 97°44'$, $b = 53°49'$, $c = 104°25'$
   (f) $a = 115°10'$, $b = 84°18'$, $c = 31°9'$

Exercises 11-6, page 250

1. (a) $a = 57°57'$, $b = 137°21'$, $C = 94°48'$
   (b) $b = 100°48'$, $A = 96°2'$, $C = 125°44'$
   (c) $c = 104°13'$, $A = 63°48'$, $B = 51°47'$
   (d) $c = 108°39'$, $A = 64°49'$, $B = 40°23'$
   (e) $c = 156°19'$, $A = 29°42'$, $B = 41°3'$
   (f) $a = 23°57'$, $b = 118°2'$, $C = 102°6'$
2. (a) $c = 9°5'$, $A = 56°30'$, $B = 115°34'$
   (b) $c = 73°41'$, $A = 130°25'$, $B = 128°26'$

Exercises 11-7, page 252

1. $c_1 = 104°19'$, $A_1 = 52°20'$, $C_1 = 124°42'$
   $c_2 = 18°10'$, $A_2 = 127°40'$, $C_2 = 15°21'$
2. $b = 15°19'$, $c = 39°0'$, $C = 98°41'$
3. $b_1 = 55°25'$, $c_1 = 81°27'$, $C_1 = 119°22'$
   $b_2 = 124°35'$, $c_2 = 162°34'$, $C_2 = 164°42'$
4. No solution
5. $b_1 = 81°15'$, $c_1 = 110°11'$, $C_1 = 119°44'$
   $b_2 = 98°45'$, $c_2 = 138°45'$, $C_2 = 142°25'$
6. $c = 88°58'$, $A = 51°44'$, $B = 139°30'$

Exercises 11-8, page 254

1. $C_n = 311°4'$, $D = 6387$ miles
2. $C_n = 297°42'$, $C_n = 225°45'$, $D = 5992$ miles
3. $C_n = 224°9'$, $D = 5832$ miles
4. $C_n = 217°1'$
5. $D = 6779.9$ miles

Exercises 12-2, page 261

	Z_n	h		Z_n	h
1.	208°12′	59°10′	**2.**	312°15′	31°13′
3.	203°48′	21°42′	**4.**	145°4′	35°33′
5.	44°41′	51°40′	**6.**	125°19′	45°53′
7.	73°12′	64°14′	**8.**	86°0′	36°40′

Exercises 12-3, page 263

1. (*a*) 3°49.5′; (*b*) 6°49.75′; (*c*) 106°18′; (*d*) 230°17.25′; (*e*) 359°8.5′; (*f*) 188°4′
2. (*a*) $8^h1^m2^s$; (*b*) $2^h41^m49.3^s$; (*c*) $5^h17^m29.07^s$; (*d*) $17^h22^m17.9^s$; (*e*) $6^h1^m2.3^s$; (*f*) $22^h8^m51.7^s$
3. (*a*) $t = 7^h8^m2^s$ A.M., $Z_n = 79°26'$; (*b*) $t = 7^h10^m41^s$ A.M., $Z_n = 84°58'$; (*c*) $t = 6^h50^m25^s$ A.M., $Z_n = 81°31'$
4. $t = 8^h23^m50^s$ A.M., $Z_n = 100°44'$
5. $t = 4^h37^m46^s$ P.M., $Z_n = 272°43'$
6. $t = 9^h10^m46^s$ A.M., $Z_n = 125°46'$
7. $t = 3^h5^m18^s$ P.M., $Z_n = 261°6'$

Exercises 12-4, page 265

1. A = E. 29°28.1′ S. **2.** $4^h37^m48^s$ A.M.
3. Summer: sunrise at $4^h37^m48^s$ A.M., sunset at $7^h22^m12^s$ P.M.; winter: sunrise at $7^h22^m12^s$ A.M., sunset at $4^h37^m48^s$ P.M.
4. (*a*) March 21: sunrise at $6^h0^m0^s$ A.M., sunset at $6^h0^m0^s$ P.M.; December 21: sunrise at $10^h19^m7^s$ A.M., sunset at $1^h40^m53^s$ P.M.; June 21: sunrise at $1^h40^m53^s$ A.M., sunset at $10^h19^m7^s$ P.M.
(*b*) March 21: A = 0°0′0″ at sunrise, A = 0°0′0″ at sunset; December 21: A = E. 66°59.5′ S. at sunrise, A = W. 66°59.5′ S. at sunset; June 21: A = E. 66°59.5′ N. at sunrise, A = W. 66°59.5′ N. at sunset
(*c*) Length of longest day: $20^h38^m14^s$; length of shortest day: $3^h21^m46^s$
6. (*a*) 10° N.; (*b*) 10°S.; (*c*) h = 13°27′, h = 33°27′; (*d*) 10° S.; (*e*) 30.25 ft.

Exercises 12-5, page 268

1. 0°	**2.** 30° N.	**3.** 50° N.	**4.** 4°6′ N.
5. 72°40′ S.	**6.** 46°58′ N.	**7.** 33°50′ N.	**8.** 12°24′ S.
9. 8°41′ S.	**10.** 0°	**11.** 7°11′	**12.** 37°33′ N.
13. 74°22′ N.	**14.** 37°24′ S.	**15.** 45°32′ N.	**16.** Impossible

Exercise 12-6, page 268

1. Z_n = 237°53′ **2.** Z_n = 125°26′, h = 13°48′
3. L_1 = 26°54′ N., L_2 = 71°19′ N., Z_1 = N. 45° W., Z_2 = N. 135° W.
4. L_1 = 25°42′ S., L_2 = 8°41′ S., Z_1 = S. 105° E., Z_2 = S. 75° E.

5. (*a*) $L_1 = 3°15'$ S., $L_2 = 43°23'$ S., $Z_1 =$ S. $25°15'$ E., $Z_2 =$ S. $154°45'$ E.
(*b*) $L_1 = 11°30'$ S., $L_2 = 62°40'$ N., $Z_1 =$ N. $41°2'$ E., $Z_2 =$ N. $138°58'$ E.
6. (*a*) $t = 4^h27^m46^s$ P.M., $Z_n = 272°44'$; (*b*) $t = 10^h7^m34^s$ A.M., $Z_n = 34°57'$
7. Within 7.6 nautical miles of the Chicago position
8. $D = 3355$ miles, $C_n = 86°49'$
9. $D = 6748.6$ miles, $C_n = 82°4'$, $L_v = 28°30'$ S., $\lambda_v = 136°14'$ E.
10. $D = 4461.7$ miles, $C_n = 302°14'$ **11.** $D = 6430.6$ miles, $C_n = 300°40'$
12. $L = 43°26'$ N., 1329.5 miles north of Honolulu
13. $169°7'$ W. **14.** $L = 66°10'$ N., $\lambda = 167°34'$ E.
15. (*a*) $L = 57°21'$ N., $\lambda = 17°34'$ W.; (*b*) $L = 44°37'$ N., $\lambda = 68°21'$ W.
16. (*a*) $4^h50^m59^s$ A.M., $7^h9^m1^s$ P.M.; (*b*) $5^h47^m56^s$ A.M., $6^h12^m4^s$ P.M.;
(*c*) 5^h50^m A.M.; 6^h10^m P.M.; (*d*) 6^h12^m A.M., $5°48^m$ P.M.
17. (*a*) $18^h28^m24^s$; (*b*) $5^h31^m36^s$
18. (*a*) $46°58'$ N.; (*b*) $41°42'$ N.; (*c*) $19°40'$ S.; (*d*) $72°40'$ S.; (*e*) $4°6'$ N.;
(*f*) $9°30'$ S.
19. (*a*) $38°30'$ N.; (*b*) $75°53'$ S.; (*c*) $74°22'$ N.; (*d*) $37°24'$ S.
20. $3^h59^m23^s$ P.M. **21.** $2^h58^m44^s$ P.M.

Exercises 13-1, page 271

1 (*a*) $\frac{1}{3}$; (*b*) 7; (*c*) $\frac{1}{\sqrt{3}}$; (*d*) 9; (*e*) $\frac{1}{25}$; (*f*) $\frac{1}{3^9}$; (*g*) $\frac{1}{35}$; (*h*) $\frac{8}{27}$; (*i*) $\frac{9}{4}$
2. (*a*) 2; (*b*) $\frac{1}{27}$; (*c*) 10; (*d*) $\frac{1}{9}$; (*e*) $-\frac{1}{2}$; (*f*) 0.1; (*g*) 1; (*h*) 0.1; (*i*) $-\frac{1}{2}$;
(*j*) $7^{-\frac{3}{2}}$; (*k*) 0; (*l*) 100; (*m*) 3; (*n*) 4; (*o*) $\frac{1}{6}$
3. (*a*) -1; (*b*) -3; (*c*) -4; (*d*) 3; (*e*) 0; (*f*) 5
4. (*a*) 5; (*b*) 3; (*c*) 1; (*d*) 3; (*e*) 0; (*f*) $\frac{4}{3}$; (*g*) $-\frac{7}{36}$; (*h*) $\frac{1}{16}$; (*i*) $\frac{4}{3}$;
(*j*) $2 \pm \sqrt{2}$; (*k*) 4.5; (*l*) $2^{\frac{42}{5}}$

Exercises 13-2, page 273

16. 36 **17.** $\frac{1}{2}$ **18.** 2 **19.** 15 **20.** 8
21. $\frac{1}{4}$ **22.** $\frac{1}{2}$ **23.** 2 **24.** 5 **25.** -5
26. 100 **27.** 0.01 **28.** $\frac{1}{9}$ **29.** $\frac{1}{7}$ **30.** 7
31. $\frac{1}{3}$ **32.** $-\frac{4}{3}$ **33.** 5 **34.** b **35.** $\frac{1}{3}$ **36.** 1

Exercises 13-3, page 275

4. 0.60206, 0.95424, 1.44716, 1.50515, 0.12494, -0.12494
5. -0.17609, 0.17609, 2.53530, 0.15052, 0.28170, 0.69897
6. (*a*) 1.47712; (*b*) 0.93305; (*c*) 0.98420; (*d*) 0.94813;
(*e*) 0.07112; (*f*) 0.21292

Exercises 13-4, page 279

1. 0 **2.** 5 **3.** 1 **4.** 0 **5.** 2 **6.** 1
7. $8 - 10$ **8.** $9 - 10$ **9.** 4 **10.** 2 **11.** $5 - 10$
12. $7 - 10$ **13.** 3 **14.** 4 **15.** $9 - 10$ **16.** $6 - 10$

Exercises 13-6, page 281

1. 1.6073 **2.** 0.4839 **3.** 3.0124 **4.** 2.0338
5. 9.3332 − 10 **6.** 7.5836 − 10 **7.** 8.9368 − 10 **8.** 5.8815 − 10
9. 8.4319 − 10 **10.** 9.2613 − 10

Exercises 13-7, page 282

1. 0.04602 **2.** 7901 **3.** 207.3 **4.** 0.5012
5. 0.009395 **6.** 997 **7.** 7.495 **8.** 2.644
9. 12.95 **10.** 0.0003527

Exercises 13-8, page 283

1. 43.39 **2.** 7.153 **3.** 1.695 **4.** 0.3311
5. 58.09 **6.** 4.29 **7.** 224.2 **8.** 0.06209

Exercises 13-9, page 284

2. (*a*) 5.019; (*b*) 147.5; (*c*) 0.000414; (*d*) 5057.7

Exercises 13-11, page 286

1. 8.54 **2.** 18.64 **3.** 0.1088 **4.** 7.595
5. 200,530 **6.** 3.141 **7.** 7.298 **8.** 0.7215
9. 0.3977 **10.** 27.28 **11.** 0.1983 **12.** 24.67
13. 1.784 **14.** 0.06567 **15.** 26.86 **16.** 1.239
17. 1.160 **18.** 0.5367 **19.** −1.255 **20.** −5.206
21. 0.007441 **22.** 1.5601, (−)1.4609, 9.0562 − 10, 2.0828
23. 46.71 **24.** 0.8646 **25.** 0.02838 **26.** 2127 lb.
27. 2283 lb. **28.** 6.269 ft. **29.** 151,370 gal. **30.** 1.01 sec.
31. 6.269 ft. **32.** Volume = 13,330, surface = 2719
33. 1051×10^7 **34.** 11,660 **35.** 834,200 **36.** 1,476,000
37. 0.608

Exercises 13-12, page 290

1. 2.367 **2.** −3.595 **3.** −1.735 **4.** −1.903
5. 1.537 **6.** 1.595 **7.** −0.1542 **8.** −0.7621
9. 6.011 **10.** 1.789 **11.** 340.8 **12.** 1.789
13. 0.4277 **14.** 0.4164 **15.** 0.1170 **16.** −0.3798
17. $x = 3.0484$, $y = 2.0484$ **18.** 17.68 **19.** $a = 0$, $b = \pm 1.317$
20. 3.96 **21.** 0.00003772 **22.** 18,360 **23.** $k = 0.126$
24. 5.5 min. **25.** $x = \dfrac{e^2 - 1}{3}$ **26.** $x = 25, -4$

Exercises 13-14, page 292

1. 222.9 **2.** 0.03734 **3.** 72.89 **4.** 0.009393
5. 24.49 **6.** 1.214 **7.** 12.38 **8.** 4.479
9. 3.068 **10.** 0.0007902 **11.** 0.3767 **12.** 0.2893
13. 0.9605 **14.** 1.787 **15.** 34.80 **16.** 67.53

17. 42.62	**18.** 2363	**19.** −4.210	**20.** −0.8605
21. −0.4616	**22.** 0.1464	**23.** 4.625	**24.** 3.506
25. 1.551	**26.** 0.03605	**27.** (*a*) 0.09318; (*b*) 168.2; (*c*) 0.4470	
28. 35.24	**29.** 31.59	**30.** 2.92 per cent	**31.** 194. ft. per sec.
33. 16,874 ft.	**34.** $x = 523$ ft., $y = 5903$ ft.		
35. 10.08 lb. per sq. in., 8.352 lb. per sq. in.			**36.** 1205 lb.
37. 4.79 sec.	**38.** (*a*) 823.7 ft.; (*b*) 49°38′; (*c*) 251.1 ft.		
39. 15.82 min.	**40.** 67.19 min.	**41.** 4.251	
43. 1547 miles	**44.** 146,700 sq. km.		

Exercises 14-1, page 302

1. 5	**2.** 7	**3.** 10	**4.** 9.1	**5.** 6.75
6. 9.62	**7.** 49.8	**8.** 340	**9.** 47.0	**10.** 0.0826
11. 3220	**12.** 0.836	**13.** 9.86	**14.** 3.08	

Exercises 14-2, page 303

1. 15	**2.** 15.8	**3.** 3530	**4.** 42.1
5. 0.001322	**6.** 1737	**7.** 9.98	**8.** 1,340

Exercises 14-3, page 304

1. 2.32	**2.** 165.2	**3.** 0.0767	**4.** 106.1	**5.** 0.000713
6. 77.5	**7.** 1861	**8.** 26.3	**9.** 1.154	**10.** 0.0419

Exercises 14-4, page 305

1. 36.7	**2.** 8.35	**3.** 0.0000632	**4.** 3400
5. 0.00357	**6.** 13,970	**7.** 1586	**8.** 0.0223
9. 0.01311	**10.** 2.36	**11.** 0.0414	**12.** 2460
13. 249	**14.** 0.275	**15.** 0.1604	**16.** 0.0977

Exercises 14-5, page 307

1. $x = 5.22$	**2.** $x = 2.30$, $y = 31.8$
3. $x = 51.7$, $y = 3370$	**4.** $x = 3.97$, $y = 9.84$, $z = 0.272$
5. $x = 0.1013$, $y = 0.0769$	**6.** $x = 1.586$, $y = 41.4$
7. $x = 106.2$, $y = 30.4$	**8.** $x = 0.1170$, $y = 0.927$
9. $x = 186$, $y = 13.42$, $z = 50.3$	

Exercises 14-6, page 308

1. 10,570	**2.** 92,200	**3.** 0.0337	**4.** 1.765	**5.** 73,100
6. 249,000	**7.** 0.002224	**8.** 0.314	**9.** 1.799	**10.** 0.1555

Exercises 14-7, page 310

1. 0.001156	**2.** 1.512	**3.** 1.015	**4.** 17.2
5. 96.1	**6.** 0.1111	**7.** 150,800	**8.** 15.32
9. 9.76	**10.** 0.00288	**11.** 144,700	**12.** 0.0267
13. 0.279	**14.** 41.3	**15.** 111.1	**16.** 3430

Exercises 14-8, page 311

1. 2.83, 3.46, 4.12, 9.43, 2.98, 0.943, 85.3, 0.252, 252, 316
2. (*a*) 231 ft.; (*b*) 0.279 ft.; (*c*) 5720 ft.
3. (*a*) 18.05 ft.; (*b*) 0.992 ft.; (*c*) 49.7 ft.

Exercises 14-9, page 312

1. 64.2	**2.** 109.2	**3.** 11.41	**4.** 0.428
5. 9.65	**6.** 0.0602	**7.** 1.525×10^5	**8.** 1.589

Exercises 14-10, page 314

2. (*a*) 0.5; (*b*) 0.616; (*c*) 0.0581; (*d*) 1; (*e*) 0.999;
(*f*) 0.0276; (*g*) 0.253; (*h*) 0.381; (*i*) 0.204; (*j*) 0.783
3. (*a*) 0.866; (*b*) 0.788; (*c*) 0.998; (*d*) 0; (*e*) 0.0393;
(*f*) 1; (*g*) 0.968; (*h*) 0.924; (*i*) 0.979; (*j*) 0.623
4. *A*. (*a*) 30°; (*b*) 61°6′; (*c*) 22°2′; (*d*) 5°44′; (*e*) 51′34″;
(*f*) 38°19′; (*g*) 3°33′; (*h*) 1°46′; (*i*) 66°56′; (*j*) 62°15′
B. (*a*) 30°; (*b*) 28°54′; (*c*) 67°58′; (*d*) 84°16′; (*e*) 89°8′;
(*f*) 51°41′; (*g*) 86°27′; (*h*) 88°13′; (*i*) 23°4′; (*j*) 27°45′
5. (*a*) 2; (*b*) 1.623; (*c*) 17.21; (*d*) 1; (*e*) 1.001;
(*f*) 36.2; (*g*) 3.95; (*h*) 2.63; (*i*) 4.90; (*j*) 1.277
6. (*a*) 1.155; (*b*) 1.27; (*c*) 1.002; (*d*) ∞; (*e*) 25.5;
(*f*) 1; (*g*) 1.033; (*h*) 1.082; (*i*) 1.021; (*j*) 1.605
7. *A*. (*a*) 30°; (*b*) 24°38′; (*c*) 36°; (*d*) 9°24′; (*e*) 0°43′; (*f*) 12°14′
B. (*a*) 60°; (*b*) 65°22′; (*c*) 54°; (*d*) 80°36′; (*e*) 89°17′; (*f*) 77°46′

Exercises 14-11, page 315

1. 0.1423, 0.515, 1.906, 0.01949, 3.55, 19.08, 1.09, 7.03, 1.942, 0.525, 51.3, 0.282, 0.0524, 0.917
2. (*a*) 13°30′; (*b*) 38°8′; (*c*) 42°37′; (*d*) 28°22′; (*e*) 3°23′; (*f*) 4°42′;
(*g*) 23°22′; (*h*) 2°28′; (*i*) 51′13″; (*j*) 20°30′; (*k*) 74°57′; (*l*) 77°55′;
(*m*) 86°38′; (*n*) 45°51′; (*o*) 50°56′
3. (*a*) 76°30′; (*b*) 51°52′; (*c*) 47°23′; (*d*) 61°38′; (*e*) 86°37′; (*f*) 85°18′
(*g*) 66°38′; (*h*) 87°32′; (*i*) 89°9′; (*j*) 69°30′; (*k*) 15°3′; (*l*) 12°5′;
(*m*) 3°22′; (*n*) 44°9′; (*o*) 39°4′

Exercises 14-12, page 316

1. 30.5	**2.** 0.360	**3.** 4.61	**4.** 24.2
5. 14.25	**6.** 16.79	**7.** 5.29	**8.** 254
9. 0.0679	**10.** 0.267	**11.** 1.349	**12.** 16.47
13. 2.033	**14.** 0.720	**15.** 4.24	**16.** 1.226
17. 0.0771	**18.** 0.0961	**19.** 38.1	**20.** 0.00319
21. 0.001091	**22.** 5.08	**23.** 0.01375	**24.** 0.0433

Exercises 14-13, page 318

1. $A = 75°$, $b = 35.46$, $c = 53.3$
2. $C = 55°$, $b = 70.7$, $a = 56.1$
3. $C = 123°12'$, $b = 2257$, $c = 2599$
4. $A = 2°47'$, $B = 87°13'$, $c = 4570$
5. $B = 35°16'$, $C = 84°44'$, $c = 138$
6. $A = 17°41'$, $C = 53°19'$, $a = 0.0751$
7. $C = 55°20'$, $b = 568$, $c = 664$
8. $b = 279$, $c = 284$, $C = 100°50'$
9. $A = 87°41'$, $C = 41°12'$, $a = 116.9$
10. Impossible
11. $B = 30°3'$, $C = 90°$, $b = 5.01$
12. $c = 123.8$, $B = 3°19'$, $C = 116°41'$
13. 1253 ft.
14. 1034.8 yd.
15. $B_1 = 66°10'$, $C_1 = 58°26'$, $c_1 = 18.6$, $B_2 = 113°50'$, $C_2 = 10°46'$, $c_2 = 4.08$
16. $B_1 = 16°43'$, $A_1 = 147°28'$, $a_1 = 35.5$, $B_2 = 163°17'$, $A_2 = 0°54'$, $a_2 = 1.04$
17. $A_1 = 70°12'$, $B_1 = 57°24'$, $b_1 = 28.79$, $A_2 = 109°48'$, $B_2 = 17°48'$, $b_2 = 10.45$
18. $A_1 = 68°47'$, $C_1 = 67°10'$, $a_1 = 6.92$, $A_2 = 23°7'$, $C_2 = 112°50'$, $a_2 = 2.91$
19. $B_1 = 45°16'$, $C_1 = 99°8'$, $c_1 = 300$, $B_2 = 134°44'$, $C_2 = 9°40'$, $c_2 = 51.1$
20. $A_1 = 51°19'$, $C_1 = 88°41'$, $c_1 = 21{,}850$, $A_2 = 128°41'$, $C_2 = 11°19'$, $c_2 = 4290$
21. $p = 3.13$; (*a*) none; (*b*) 2; (*c*) 1

Exercises 14-14, page 320

1. $A = 31°20'$, $B = 58°40'$, $c = 23.7$
2. $A = 33°9'$, $B = 56°51'$, $c = 499$
3. $A = 45°$, $B = 45°$, $c = 18.67$
4. $A = 41°2'$, $B = 48°58'$, $c = 153.8$
5. $A = 39°30'$, $B = 50°30'$, $c = 44$
6. $A = 30°37'$, $B = 58°23'$, $c = 82.5$
7. $A = 65°$, $B = 25°$, $c = 55.2$
8. $A = 67°23'$, $B = 22°37'$, $c = 13$
9. $A = 3°42'$, $B = 86°18'$, $c = 4.8$

Exercises 14-15, page 321

1. $A = 119°54'$, $B = 31°6'$, $c = 52.6$
2. $A = 49°4'$, $C = 79°7'$, $b = 104.1$
3. $A = 55°2'$, $B = 40°21'$, $c = 285$
4. $B = 39°16'$, $C = 78°44'$, $a = 3.21$
5. $A = 100°57'$, $C = 33°3'$, $b = 19.8$
6. $A = 46°26'$, $C = 6°24'$, $b = 7.43$
7. $A = 121°4'$, $C = 2°26'$, $b = 0.0828$
8. $A = 77°12'$, $B = 43°30'$, $c = 15$
9. $B = 13°22'$, $C = 28°17'$, $a = 7420$
10. 10 and 4.68
11. 4.93 miles

Exercises 14-16, page 323

1. $A = 106°47'$, $B = 46°53'$, $C = 26°20'$

2. $A = 52°26'$, $B = 59°23'$, $C = 68°12'$

3. $A = 44°42'$, $B = 49°37'$, $C = 85°40'$

4. $A = 27°21'$, $B = 143°8'$, $C = 9°32'$

5. $A = 49°12'$, $B = 37°36'$, $C = 93°12'$

6. $A = 83°42'$, $B = 59°22'$, $C = 36°56'$

Exercises 14-17, page 323

1. (*a*) 0.785; (*b*) 1.047; (*c*) 1.571; (*d*) 3.14; (*e*) 2.09; (*f*) 2.36; (*g*) 0.393; (*h*) 3.49; (*i*) 52.4

2. (*a*) 60°; (*b*) 135°; (*c*) 2.5°; (*d*) 210°; (*e*) 1200°; (*f*) 176.4°

3. (*a*) 0.01745; (*b*) 0.0002909; (*c*) 0.00000485; (*d*) 0.1778; (*e*) 3.152; (*f*) 5.24

4. (*a*) 5°44′; (*b*) 143°15′; (*c*) 91°40′; (*d*) 343°46′